

Derived

The Calabi-Yau Auslander-Iyama Correspondence

(joint work with Fernando Muro)

Fix  $k$ : perfect field (e.g.  $\text{char } k = 0$  or  $k = \bar{k}$ ) &  $d \geq 1$ Thm (A) (Muro 2022  $d=1$ , J-Muro 2022<sup>+</sup>  $d \geq 1$ , J-Muro 2025<sup>+</sup>)

There is a 1:1 correspondence between the following:

- (1) Quasi-iso classes of dg algebras  $A$  s.t.
- (a)  $H^0(A)$  is a basic fin. dim. alg.
  - (b)  $A \in \text{perf}(A)$  is  $d\mathbb{Z}$ -cluster tilting ( $\Rightarrow A \cong A[d]$  in  $\text{perf}(A)$ )
  - (c)  $A[n] \cong DA$  in  $\mathcal{D}(A^e)$ ,  $n \in d\mathbb{Z}$   $A$  is bimodule right  $n$ -CY

- (2) Equivalence classes of pairs  $(\Lambda, I)$  s.t.
- (a)  $\Lambda$  is a basic Frobenius alg.
  - (b)  $I \in \text{mod}(\Lambda^e)$   $\otimes_{\Lambda^e}$ -invertible s.t.  $\exists [\gamma] \in \text{Ext}_{\Lambda^e}^{d+2}(\Lambda, I)$

$$\gamma: 0 \rightarrow I \rightarrow \underbrace{P_{d+1} \rightarrow P_d \rightarrow \dots \rightarrow P_1 \rightarrow P_0}_{\text{proj}(\Lambda^e)} \rightarrow \Lambda \rightarrow 0$$

*twisted  $(d+2)$ -periodic*

$$\Lambda := \bigoplus_{d \in d\mathbb{Z}} \tau^d \Lambda, \text{ where } \tau \in \text{Aut}(\Lambda) \text{ satisfies } I \cong_{\Lambda^e} \tau I$$

- (c)  $\exists \varphi: \Lambda(n) \xrightarrow{\sim} D\Lambda$ ,  $n \in d\mathbb{Z}$ , in  $\text{mod}^{\mathbb{Z}}(\Lambda^e)$  s.t.

*Universal Mumey Product  $\in \text{HH}^{d+2, -d}(\Lambda, \Lambda)$*

$$\Delta_{\varphi}(\{u_{d+2}^{\mathbb{Z}}\}) = 0 \text{ in } \text{HH}^{d+1, -d}(\Lambda, \Lambda)$$

*Batalin-Vilkoviski operator*

The correspondence (1)  $\rightarrow$  (2) is given by  $A \mapsto (H^0(A), H^{-d}(A))$

Moreover, for  $A$  as in (1),  $\text{perf}(A)$  admits a unique dg enhancement.

Def (Iyama-Yoshino 2008, Geiss-Keller-Oppermann 2013)

$H^*(A)$   
degree-wise  
fin. dim.  $A \in \text{perf}(A)$ : basic <sup>fixed</sup> is  $d\mathbb{Z}$ -cluster-tilting if  $A \cong A[d]$  and

$$\text{add}(A) = \{ X \in \text{perf}(A) \mid \forall 0 < i < d \text{ Hom}_A(A, X[i]) = 0 \}$$

$$= \{ X \in \text{perf}(A) \mid \forall 0 < i < d \text{ Hom}_A(X, A[i]) = 0 \}$$

Remark  $A \in \text{perf}(A)$  is  $1\mathbb{Z}$ -CT  $\Leftrightarrow \text{add}(A) = \text{perf}(A)$

Remark  $A \in \text{perf}(A)$  is  $d\mathbb{Z}$ -CT  $\Rightarrow H^*(A) = H^{d\mathbb{Z}}(A)$   
 $A[n] \cong DA$  in  $\mathcal{D}(A^e) \Rightarrow H^*(A)(n) \cong DH^*(A)$  as <sup>graded</sup>  $H^*(A)$ -bimodules

Thm (Dugas 2012, Green-Susashall-Solberg 2003 + GKO 2013  
 + Amiot 2007  $d=1$  / Liu 2019  $d \geq 1$  + Hanihara 2020)

$A \in \text{perf}(A)$ : basic  $d\mathbb{Z}$ -CT  $\Rightarrow (H^0(A), H^{-d}(A))$  satisfies conditions (2a) & (2b) in Thm A.

Remark  $A \in \text{perf}(A)$ :  $d\mathbb{Z}$ -CT  $[d] \subset \text{add}(A) \xrightarrow{\sim} \text{proj}(H^0(A)) \cup \bigoplus_{H^0(A)} H^d(A)$

Example  $A = k\langle z^{\pm} \rangle$ ,  $|z| = -d$ . Easy:  $A \in \text{perf}(A)$  is  $d\mathbb{Z}$ -CT,  $H^0(A) = k$   
 $A[d] \cong DA$  in  $\mathcal{D}(A^e)$

Example  $\Pi := (d+1)$ -CY completion of a  $d$ -rep.-finite  $d$ -hereditary alg.  
 $A := \text{REnd}(\Pi)$  in  $\text{cosing}(\Pi) := \text{perf } \Pi / \text{pvd } \Pi$   
 $A \in \text{perf}(A)$ :  $d\mathbb{Z}$ -CT and  $H^0(A) \cong H^0(\Pi)$ :  $(d+1)$ -preproj. alg.  
 $A[d] \cong DA$  in  $\mathcal{D}(A^e)$

( $d=1$ )  $\Pi :=$  derived 2-preprojective alg. of ADE quiver  $H^0(\Pi)$ : preproj. alg.  
 $k = \mathbb{C}$   $\text{perf}(A) = \underline{\text{CM}}$  (corresponding ADE surface sing.)

( $d=2$ )  $\Pi :=$  Ginzburg 3-CY dg alg of a self-injective QP.  $H^0(\Pi)$ : Jacobian algebra  
 $\text{perf}(A)$  corresponding 2-CY Amiot cluster category.

$A$ : dg algebra with the following properties

- $H^*(A) = H^{d\mathbb{Z}}(A)$ , i.e.  $H^i(A) \neq 0 \Rightarrow i \in d\mathbb{Z}$
- $A$  is locally proper, i.e.  $\forall i \in \mathbb{Z}$   $H^i(A)$  is finite-dimensional
- $H^0(A)$  is a basic fin. dim. alg.
- $\exists \varphi \in H^d(A) \cong \text{Hom}_{\mathcal{D}(A)}(A, A[d])$  invertible

e.g.  $A \in \text{perf}(A)$  is basic  $d\mathbb{Z}$ -CT.

$$\Lambda := H^0(A), \quad \Lambda = \bigoplus_{\mathbb{Z}} \langle 2^{\pm} \rangle * H^0(A) \cong H^*(A), \quad |2| = -d, \quad \sigma: x \mapsto \varphi^{-1} x \varphi$$

Remark  $\Lambda^{-d} \cong H^{-d}(A)$  as  $\Lambda$ -bimodules

Choose: minimal  $A_0$ -algebra model  $(H^*(A), m^A)$  for  $A$ .

Notice:  $m_{i+2}^A \neq 0 \Rightarrow i \in d\mathbb{Z}$  since  $H^*(A) = H^{d\mathbb{Z}}(A)$  &  $|m_{i+2}^A| = -i$

$$\rightsquigarrow (H^*(A), m^A) = (\Lambda, m_{d+2}^A, m_{2d+2}^A, m_{3d+2}^A, \dots)$$

$m_{d+2}^A \in C^{d+2, -d}(\Lambda, \Lambda)$  is a Hochschild cocycle

$\rightsquigarrow \{m_{d+2}^A\} \in \text{HH}^{d+2, -d}(\Lambda, \Lambda)$ : Univ. Massey Product of length  $d+2$

$j: \Lambda \hookrightarrow \Lambda$  induces a map

$$j^*: \text{HH}^{d+2, -d}(\Lambda, \Lambda) \longrightarrow \text{HH}^{d+2, -d}(\Lambda, \Lambda)$$

$$\{m_{d+2}^A\} \longmapsto j^* \{m_{d+2}^A\} \in \text{Ext}_{\Lambda^e}^{d+2}(\Lambda, \Lambda_r)$$

$\uparrow$  restricted UMP of length  $d+2$

Thm (J-Muro 2022<sup>+</sup>) For  $A$  as above, TFAE:

(1)  $A \in \text{perf}(A)$  is (basic)  $d\mathbb{Z}$ -CT

(2)  $j^* \{ \omega_{d+2}^A \} \in \text{Ext}_{\Lambda}^{d+2}(\Lambda, \Lambda_\sigma)$  exhibits an isomorphism

$$j^* \{ \omega_{d+2}^A \} : \Omega_{\Lambda^e}^{d+2}(\Lambda) \xrightarrow{\sim} \Lambda_\sigma \quad \text{in } \underline{\text{mod}} \Lambda^e$$

Slogan The restricted UMP of length  $d+2$  witnesses the  $d\mathbb{Z}$ -CT property of the dg algebra  $A$ .

Thm (B) (J-Muro 2022<sup>+</sup>)  $A$ : dg alg. s.t.  $H^*(A) = H^{d\mathbb{Z}}(A)$  and

$$\text{HM}^{p+2, -p}(H^*(A), \{ \omega_{d+2}^A \}) = 0 \quad \text{for } p > d$$

↑ Hochschild-Massey cohomology

If  $B$  is a dg algebra with  $H^*(B) \cong H^*(A)$  and

$$\{ \omega_{d+2}^B \} = \{ \omega_{d+2}^A \} \in \text{HH}^{d+2, -d}(H^*(A), H^*(A))$$

then  $B$  is quasi-isomorphic to  $A$ .

Remark For  $d=1$  and  $\text{HH}^{3, -1}(H^*(A), H^*(A)) = 0$ , Thm B recovers Kadeishvili's Intrinsic Formality Theorem.

$\text{HM}(H^*(A), \{ \omega_{d+2}^A \})$  is the cohomology of

↳ Hochschild-Massey complex

$$\text{CM}^{p, q}(H^*(A), \{ \omega_{d+2}^A \}) := \text{HH}^{*, *}(H^*(A), H^*(A)) \quad p \geq 2, q \in \mathbb{Z}$$

$d_{\text{HM}} : x \longmapsto [\{ \omega_{d+2}^A \}, x]$  of bidegree  $(d+1, -d)$  (if  $\text{char } k \neq 2$ )

Rmk  $d_{HM} \circ d_{HM} = 0$  uses  $Sg(\{m_{d+2}^A\}) = 0$

Rmk  $CM(H^*(A), \{m_{d+2}^A\})$  is the  $(d+1)$ -st page of an extended Bousfield-Kan spectral sequence (Muro 2020  $d=1$ )

Aim Analogues of  $\{m_{d+2}^A\}$  and Thm B for bimodules?

$\Lambda$ : graded algebra Fix:  $M$  a graded  $\Lambda$ -bimodule

$$C_{\Lambda}^{n,r}(M, M) := \bigoplus_{p+q=n} C_{\Lambda}^{p,q,r}(M, M) \quad n \geq 0, r \in \mathbb{Z}, \text{ where}$$

$$C_{\Lambda}^{p,q,r}(M, M) := \text{Hom}_{\Lambda}(\Lambda^{\otimes p} \otimes M \otimes \Lambda^{\otimes q}, M(r))$$

$$\rightsquigarrow H^{*,*}(C_{\Lambda}(M, M), -d_{\text{Bim}}) \cong \text{Ext}_{\Lambda}^{*,*}(M, M)$$

$C_{\Lambda}(M, M)$  is naturally a  $C(\Lambda, \Lambda)$ -bimodule

We introduce the bimodule Hochschild cochain complex

$$C(\Lambda | M) := \text{cocone} \left( C(\Lambda, \Lambda) \xrightarrow{\delta} C_{\Lambda}^e(M, M) \right)$$

$\cup_{d_{\text{Hoch}}}$   $\cup_{-d_{\text{Bim}}}$

$$C^{n,r}(\Lambda | M) := C^{n,r}(\Lambda, \Lambda) \oplus C_{\Lambda}^{n-1,r}(M, M)$$

It has the bidegree  $(1,0)$  differential

$$d_{\Lambda | M} := \begin{pmatrix} d_{\text{Hoch}} & 0 \\ \delta & d_{\text{Bim}} \end{pmatrix} \quad \delta : c \longmapsto s(\text{id}_M \cdot c - (-1)^{|c|} c \cdot \text{id}_M)$$

$HH^{*,*}(\Lambda | M) := H^{*,*}(C(\Lambda | M))$  is also a Gerstenhaber alg.

! bimodule Hochschild cohomology

By construction, there is a triangle of bigraded vect. spaces

$$C_{\mathbb{N}e}(M, M)[-1] \rightarrow C(\mathbb{N} | M) \rightarrow C(\mathbb{N}, \mathbb{N}) \xrightarrow{\delta} C_{\mathbb{N}e}(M, M)$$

Remark  $C(\mathbb{N}, \mathbb{N})$  is to the endomorphism operad  $\text{End}(\mathbb{N})$   
 what  $C(\mathbb{N} | M)$  is to the linear end. operad  $\text{End}(\mathbb{N}, M)$

$A$ : dg. alg. st.  $\mathbb{N} := H^*(A) = \text{HdZ}(A)$

Consider now  $M$  a dg  $A$ -bimodule s.t.  $H^*(M) = \text{HdZ}(M)$   
 (e.g.  $M = DA$  or  $M = A[n]$  with  $n \in \mathbb{Z}$ )

Choose a minimal  $A_\infty$ -algebra model  $(H^*(A), \mathfrak{m}^A)$  of  $A$   
 and a compatible minimal  $A_\infty$ -bimodule model

$(H^*(M), \mathfrak{m}^M)$  of  $M$ ,  $\mathfrak{m}_n^M = (\mathfrak{m}_{p,q}^M)_{p+q=n}$   $|\mathfrak{m}_n^M| = 2-n$

$$\mathfrak{m}_{p,q}^M : \underbrace{H^*(A) \otimes \dots \otimes H^*(A)}_p \otimes \underbrace{H^*(M)}_1 \otimes \underbrace{H^*(A) \otimes \dots \otimes H^*(A)}_q \rightarrow H^*(M)$$

$$\mathfrak{m}_n^{A \times M} := \mathfrak{m}_n^A + \mathfrak{m}_n^M \in C^{n, 2-n}(H^*(A) | H^*(M))$$

As before,  $\mathfrak{m}_{i+2}^M \neq 0 \Rightarrow i \in \mathbb{Z}$  since  $H^*(M) = \text{HdZ}(M)$

$\mathfrak{m}_{d+2}^{A \times M} \in C^{d+2, -d}(H^*(A) | H^*(M))$  is a cycle

$\rightsquigarrow \{\mathfrak{m}_{d+2}^{A \times M}\} \in \text{HH}^{d+2, -d}(H^*(A) | H^*(M))$ ,  $S_q(\{\mathfrak{m}_{d+2}^{A \times M}\}) = 0$   
 $\uparrow$  bimodule UMP of length  $d+2$

$CM_{\mathbb{N}e}^{n,r}(H^*(M), \{\mathfrak{m}_{d+2}^{A \times M}\}) := \text{Ext}_{\mathbb{N}e}^{n,r}(H^*(M), H^*(M))$   $n \geq 1, r \in \mathbb{Z}$   
 $\uparrow$  bimodule Murray complex

with the bidegree  $(d+1, -1)$  differential

$$d_{\text{Bim HM}} : x \longmapsto [ \{ \omega_{dt_2}^{A \times M} \}, x ]$$

↖ Bracket of  $HH^{*,*}(H^*(A) | H^*(M))$

$$EM_{\mathbb{A}^e}^{*,*}(H^*(M), \{ \omega_{dt_2}^{A \times M} \}) := H^{*,*}(CM_{\mathbb{A}^e}(M, \{ \omega_{dt_2}^{A \times M} \}))$$

↑ bimodule Massey cohomology

Thm (B) for bimodules (J-Mun 2025+)

$A$  : dg alg. st.  $H^*(A) = H^{d\mathbb{Z}}(A)$       such that  
 $M$  : dg  $A$ -bim. st.  $H^*(M) = H^{d\mathbb{Z}}(M)$

$$EM_{\mathbb{A}^e}^{p+1, -p}(H^*(M), \{ \omega_{dt_2}^{A \times M} \}) = 0 \quad \forall p > d$$

If  $N$  is a dg  $A$ -bim s.t.  $H^*(N) \cong H^*(M)$  and

$$\{ \omega_{dt_2}^M - \omega_{dt_2}^N \} = 0 \in \text{Ext}_{\mathbb{A}^e}^{d+1, -d}(H^*(M), H^*(M))$$

then  $N$  is quasi-isomorphic to  $M$ .

Prop (J-Mun 2025+)  $A$  : dg alg. st.  $H^*(A) = H^{d\mathbb{Z}}(A)$

Suppose  $\exists \varphi : H^*(A)(n) \xrightarrow{\sim} DH^*(A)$  for some  $n \in \mathbb{Z}$

Then, TFAE :

↖ diagonal  $A$ -bimodule

$$(1) \quad \{ \omega_{dt_2}^{A[0]} - \omega_{dt_2}^{DA[-n]} \} = 0 \in \text{Ext}_{\mathbb{A}^e}^{d+1, -d}(H^*(A), H^*(A))$$

|||

$$(2) \quad \Delta_{\varphi}(\{ \omega_{dt_2}^A \}) = 0 \in HH^{d+1, -d}(H^*(A), H^*(A))$$

↑ algebra UMP of length  $dt_2$

Thm (J-Muro 2025<sup>+</sup>)  $\Lambda$ : graded algebra

$$HH^{*,*}(\Lambda | \Lambda) \xrightarrow[\text{iso}]{\exists K} HH^{*,*}(\Lambda)[\varepsilon] \quad \varepsilon^2 = 0 \quad |\varepsilon| = (1, 0)$$

If  $\Lambda = H^*(A) = H^{dz}(A)$  for a dg algebra  $A$ , then

$$K: \left\{ \mathcal{M}_{d+2}^{A[0]} - \mathcal{M}_{d+2}^{DA[-n]} \right\} \longmapsto \Delta_\varphi(\{ \mathcal{M}_{d+r}^A \}) \cdot \varepsilon$$

Remark

$$\begin{array}{ccc}
 C(\Lambda | \Lambda) & \xrightarrow{\Psi} & C(\Lambda | \Lambda(n)) \\
 \gamma \downarrow & = & \downarrow \theta \\
 C(\Lambda | \Lambda) & \xleftarrow{\Phi} & C(\Lambda | D(\Lambda(n)))
 \end{array}$$

We identify  $\mathcal{M}^{A \times D(A[n])} = \gamma(\mathcal{M}^{A \times A[0]})$

$\uparrow$  induced by iso  $\varphi: \Lambda(n) \xrightarrow{\sim} D\Lambda$

$$\Psi(\mathcal{M}^{A \times A[0]}) = \mathcal{M}^{A \times A[n]}, \quad \theta(\mathcal{M}^{A \times D(A[n])}) = \mathcal{M}^{A \times D(A[n])}$$

The Batalin-Vilkoviski operator (assoc. to  $\varphi: \Lambda(n) \xrightarrow{\sim} D\Lambda$ )

$\Delta = \Delta_\varphi: C^{m,r}(\Lambda | \Lambda) \longrightarrow C^{m-1,r}(\Lambda, \Lambda)$  determined by

$$\langle S^n \Delta(c)(x_1, \dots, x_{m-1}), x_m \rangle$$

$$= \sum_{i=1}^m (-1)^{i(m-1)+ks} \langle S^n c(x_i, \dots, x_m, x_1, \dots, x_{i-1}), 1_A \rangle_\varphi$$

$$\langle S^n x, y \rangle_\varphi := \varphi(S^n x)(y) \quad x, y \in \Lambda$$