UNIVERSITY
OF COLOGNE

BRACKETS, TREES AND THE BORROMEAN RINGS

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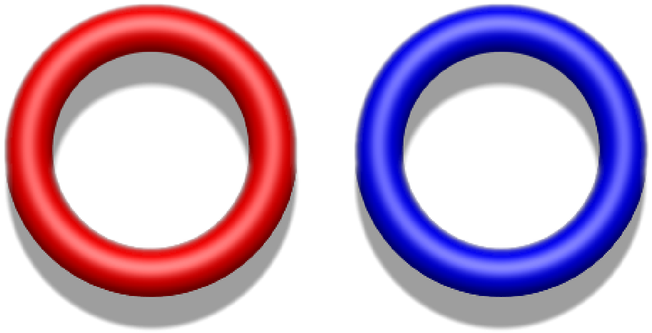
A bird's eye view on algebraic structures

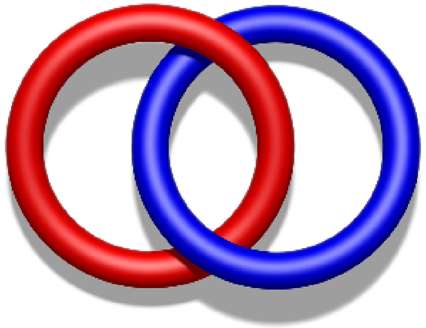
Mathematical objects	→	Algebraic structures
X : topological space		$\pi_k(X,x)$: homotopy groups
X : smooth manifold		$\Omega(X)$: algebra of differential forms
V : complex variety		$\mathbb{C}[V]$: coordinate ring
G : Lie group		$\mathrm{Lie}(G)$: Lie algebra

Properties of the mathematical objects should be reflected in properties of the algebraic structures

What differentiates these configurations of each other?

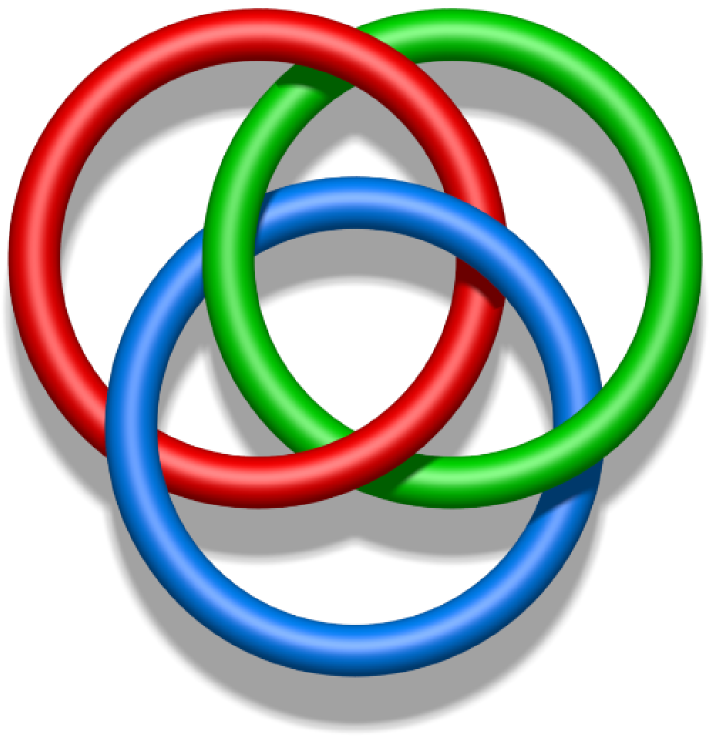


Hopf link



The Hopf link is, well, linked!

Consider now the Borromean rings:



What differentiates them from three unlinked circles?

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01

BRACKETS AND TREES

The combinatorics of the associativity equation

The familiar associativity equation

$$(ab)c = a(bc)$$

is the source of rich combinatorics:

1 : (ab)

2 : $((ab)c)$, $(a(bc))$

5 : $(((ab)c)d)$, $((a(bc))d)$, $(a((bc)d))$

$(a(b(cd)))$, $((ab)(cd))$

14 : $((((ab)c)d)e)$, $(((a(bc))d)e)$, ...

Thm (Catalan 1838)

There are precisely

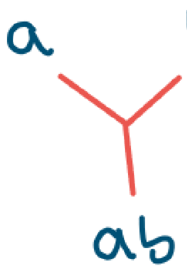
$$C_n := \frac{1}{n+1}\binom{2n}{n}$$

different ways to correctly parenthesise a word on $n+1$ letters.

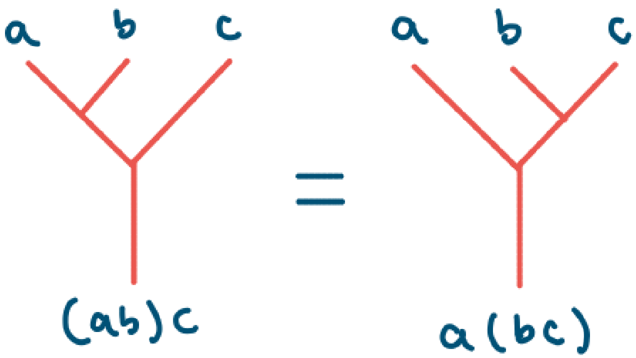
$$C_0 := 1, \qquad C_{n+1} := \sum_{k=0}^{n} C_k C_{n-k}$$

From associativity to dendrology

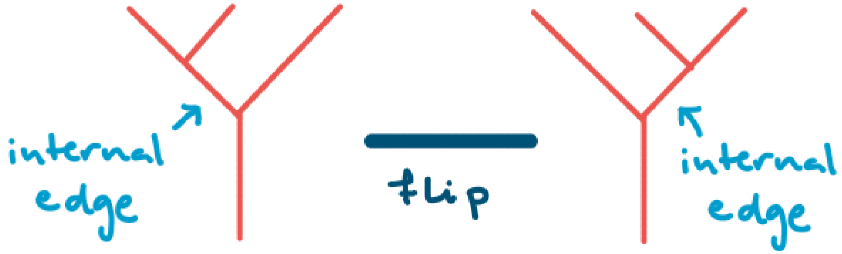
Depict the binary operation as a tree



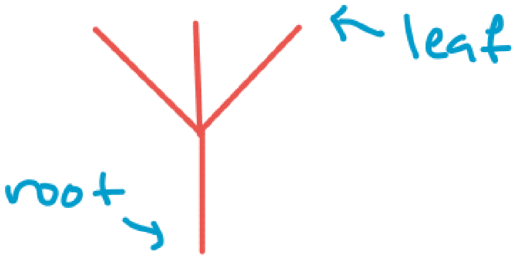
The associativity eq'n becomes



This suggest the local move:

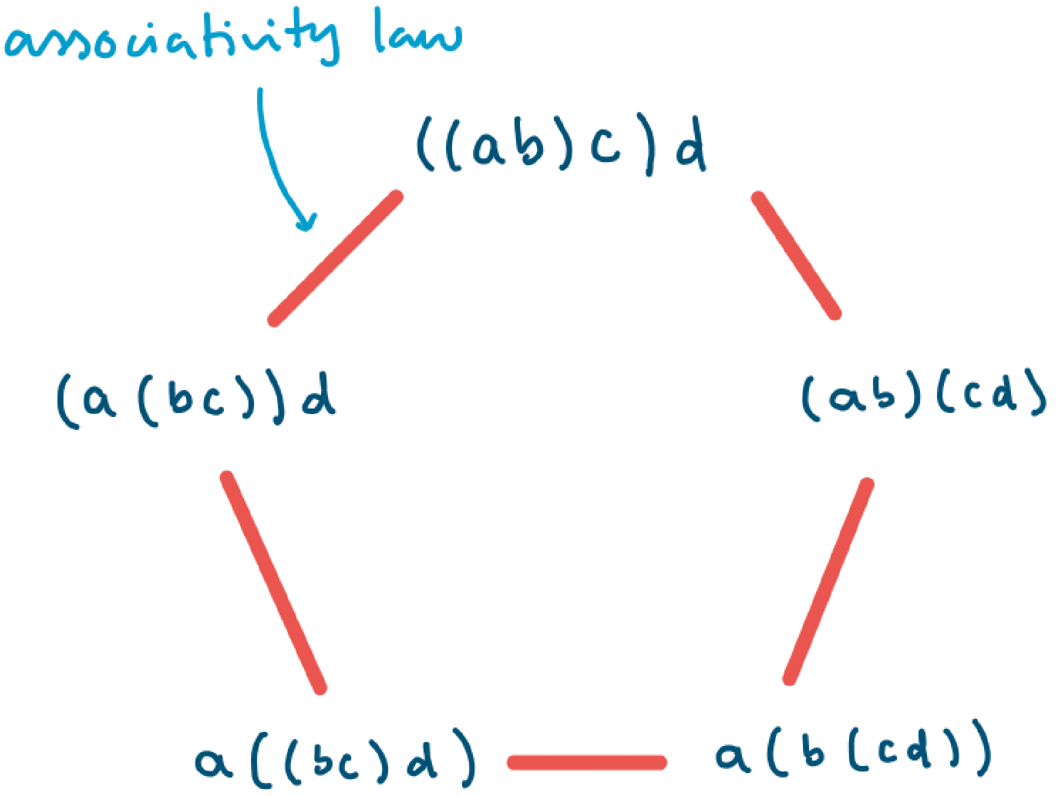


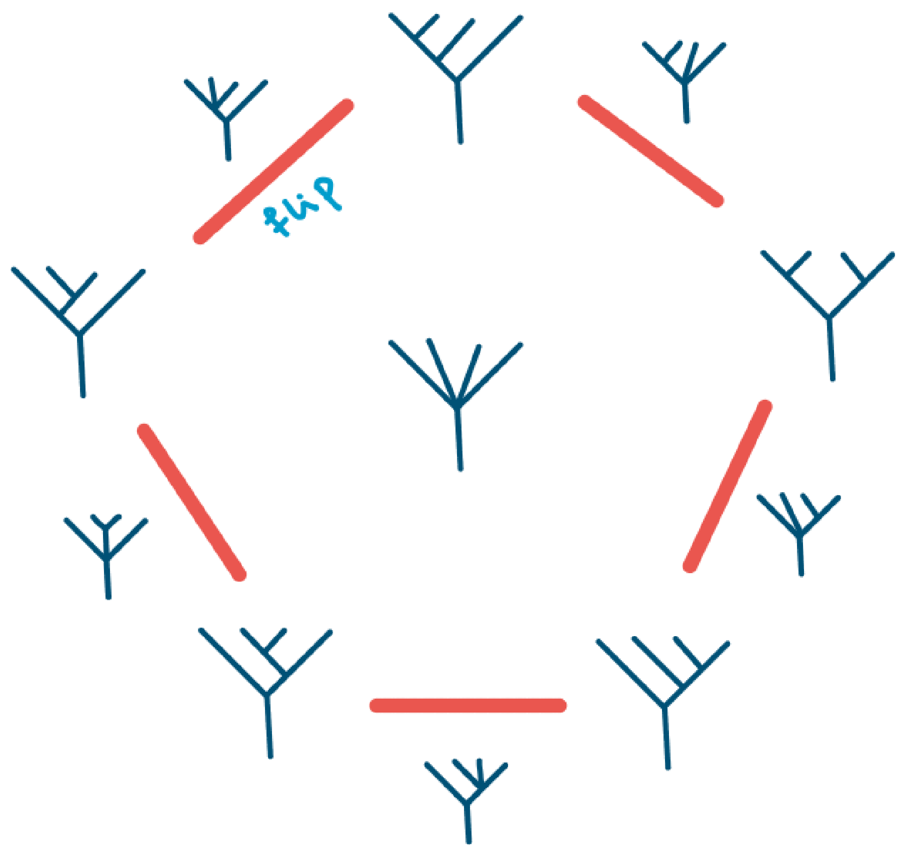
Contracting the internal edges:

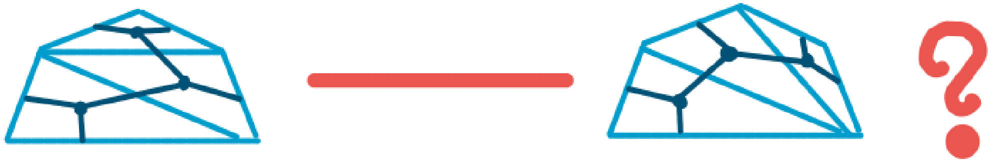


The missing planar tree with three leaves

From associativity to dendrology II







The Associahedron: "A mythical polytope"

Thm (Tamari 1951, Stasheff 1963, Loday 2004)

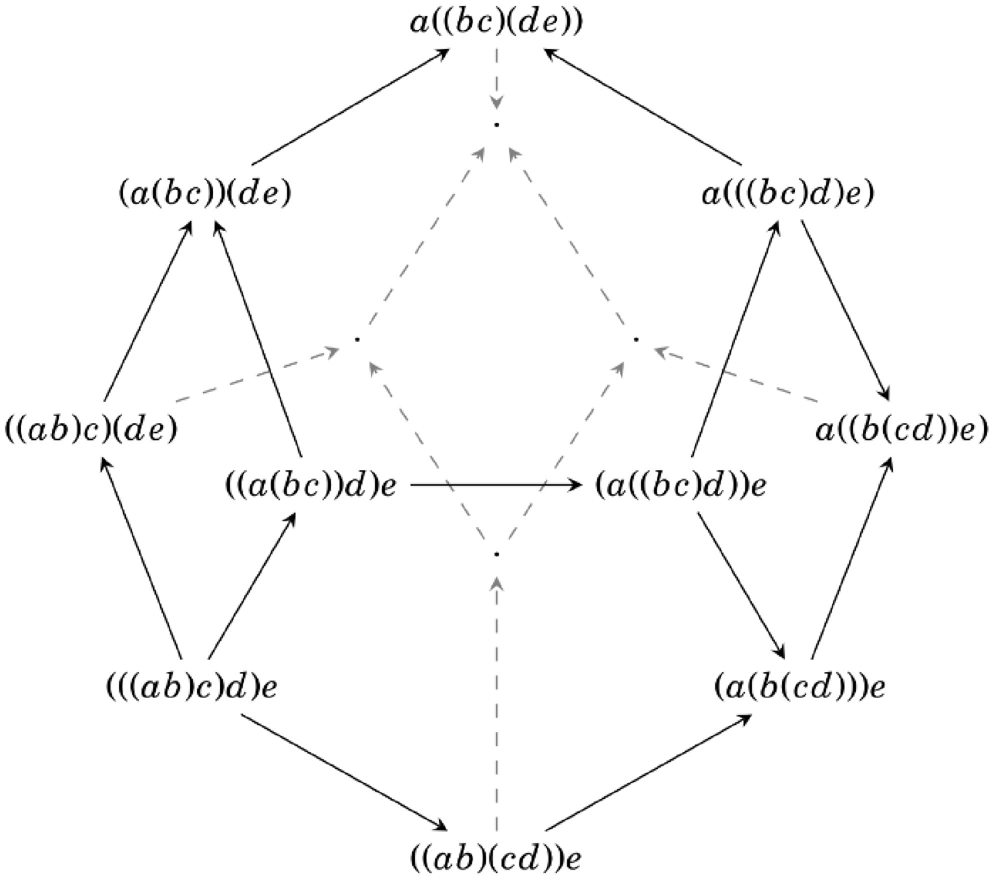
$\exists$ convex polytope K_{n+1} of dim $n-1$ s.t.

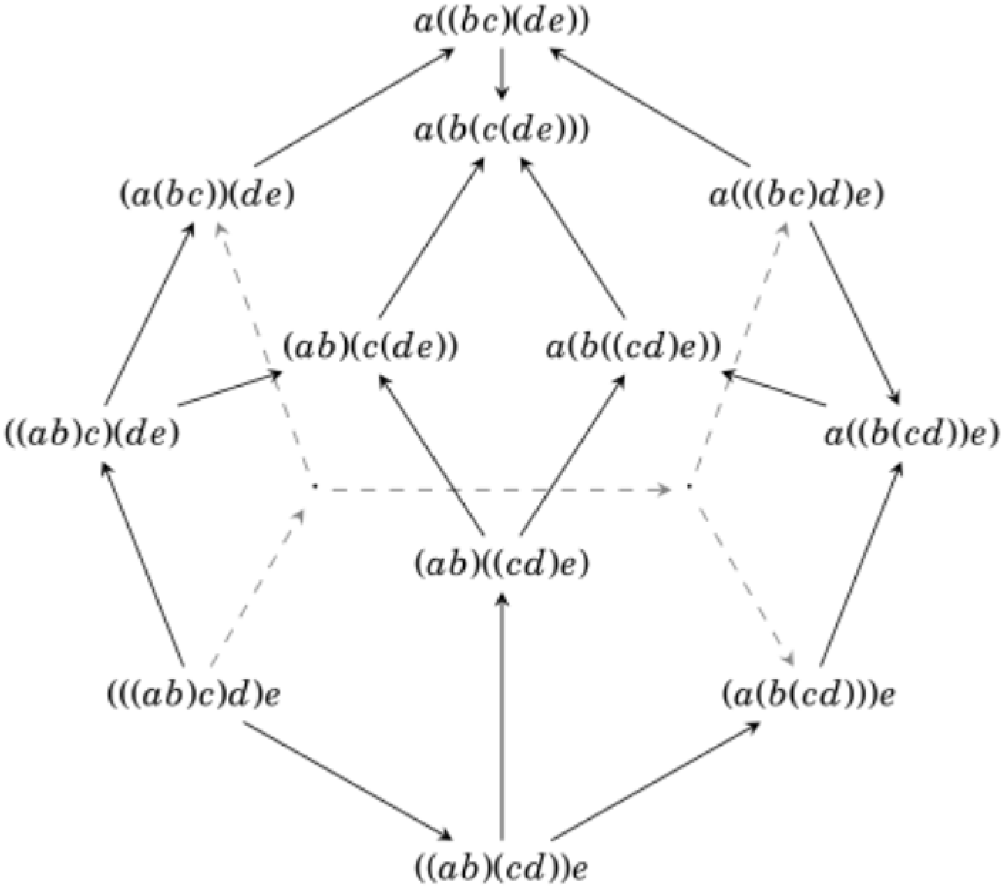
$$\left\{ \begin{array}{c} k\text{-dim cells} \\ \text{of } K_{n+1} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{planar rooted trees with} \\ n+1 \text{ leaves and } n-k \text{ int. vertices} \end{array} \right\}$$

called the Associahedron of dim $n-1$

Rmk Edges of the associahedron correspond to flips!

The Associahedron K_5

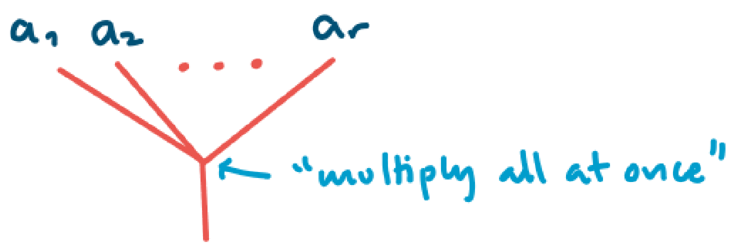




Planar trees as multi-operations

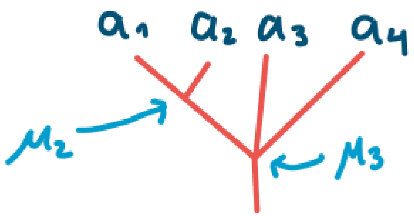
Rooted corolla with r leaves

$\rightsquigarrow$ operation with r inputs



$\mu_r(a_1, a_2, \dots, a_r)$

Combinations of these yield new multi-operations

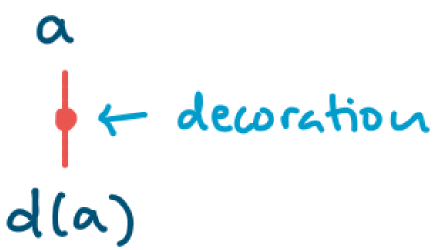


$\mu_3(\mu_2(a_1, a_2), a_3, a_4)$

We also consider a unary operation

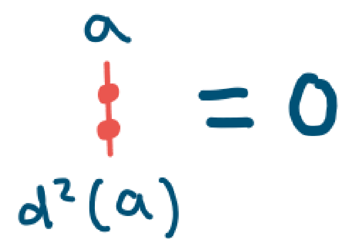
$$a \longmapsto \mu_1(a) =: d(a)$$

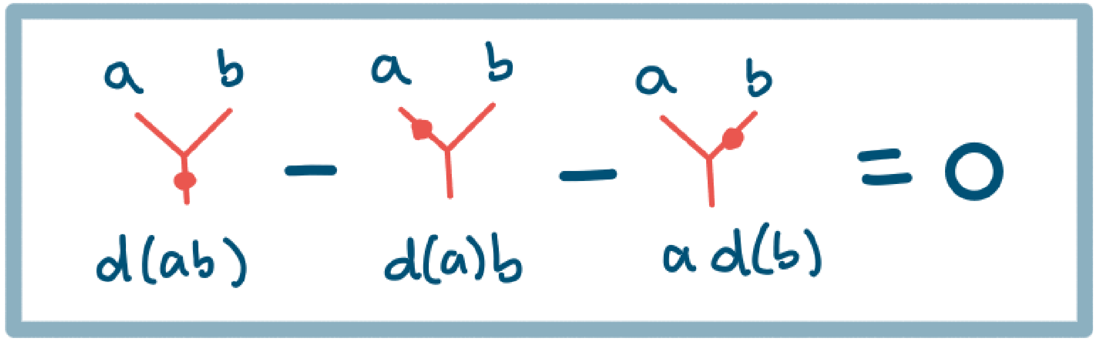
corresponding to the tree



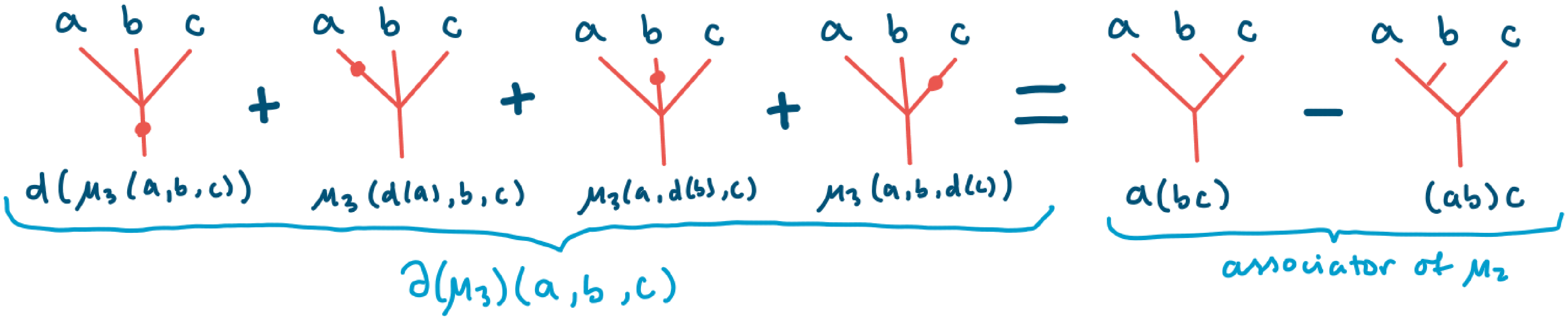
Rmk The ● indicates that something happens as a flows through the tree

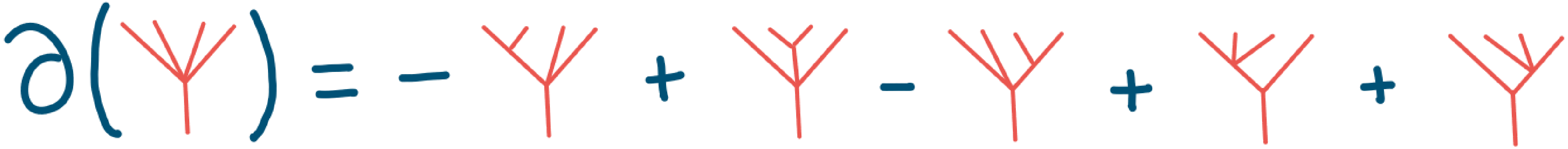
The A_∞-equations (Stasheff 1963)





Leibniz Rule





etc.

A simple example of an A_∞-structure

$\varphi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ linear map s.t. $\varphi^\ell = 0$ $\ell \geq 3$ fixed

$(\mathbb{C}^n, \varphi)$: representation of $\mathbb{C}[x]/(x^\ell) =: A$

JNF $\varphi \cong J_0(n_1) \oplus \cdots \oplus J_0(n_k)$

$n = n_1 + \cdots + n_k$

Simple(st) representation : $S = (\mathbb{C}, \boxed{0})$

$$J_0(m) = \begin{pmatrix} \boxed{0} & 1 & & 0 \\ & \boxed{0} & \ddots & \\ & & \ddots & 1 \\ 0 & & & \boxed{0} \end{pmatrix}$$ is "built" from S

The "building procedure" is encoded in the Youeda algebra:

$$\left.\begin{array}{l} \mathrm{Ext}^*_A(S,S) \cong \mathbb{C}[\varepsilon, t]/(\varepsilon^2) \\ |\varepsilon| = 1 \quad |t| = 2 \end{array}\right\}$$ independent of ℓ

$\mu_1 = 0 \qquad \mu_2 =$ Yoneda product

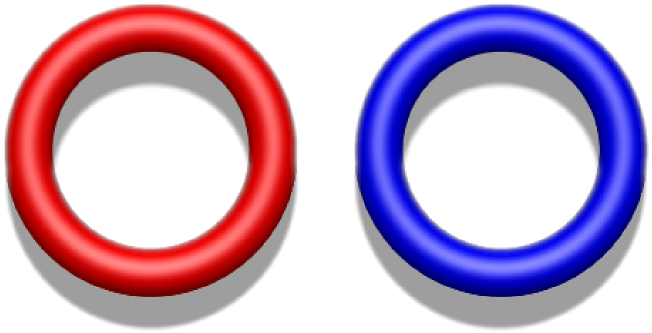
$$\boxed{\mu_\ell(\varepsilon, \ldots, \varepsilon) = t}$$

$\mu_k = 0$ for $k \neq 2, \ell$

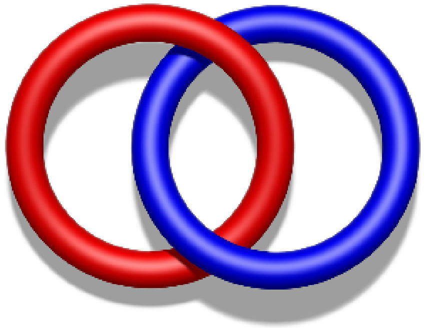
02

BACK TO THE BORROMEAN RINGS

What differentiates these configurations of each other?

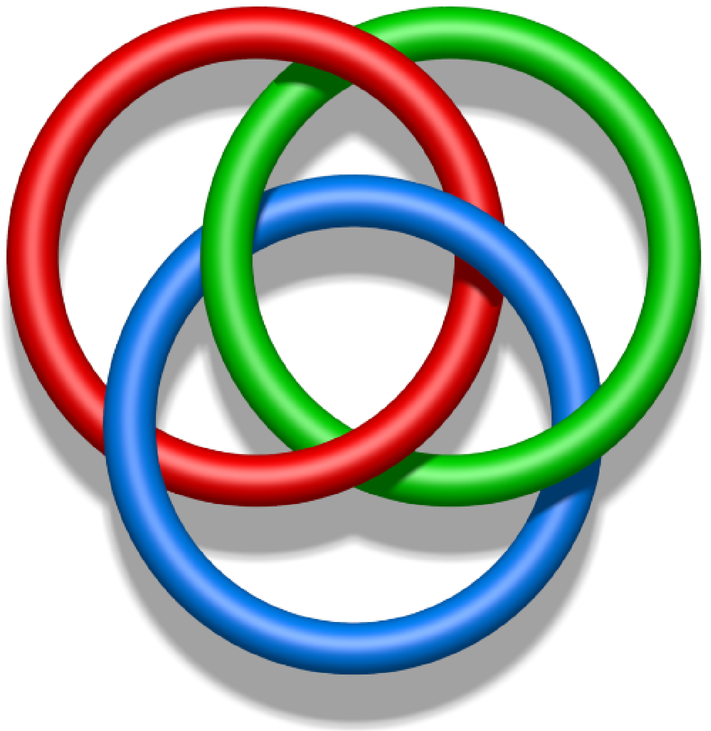


Hopf link



The Hopf link is, well, linked!

Consider now the Borromean rings



What differentiates them from three unlinked circles?

Image Credits: Jim.belk, Public domain, via Wikipedia Commons

More on the Hopf link

$$X := S^3 \setminus \text{Hopf link}$$

The singular cohomology of X is

$$H^0(X;\mathbb{R}) = \mathbb{R}$$

$$H^1(X;\mathbb{R}) = \mathbb{R}\alpha \oplus \mathbb{R}\beta$$

$$H^2(X;\mathbb{R}) = \mathbb{R}$$

with the cup product $\mu_2^X(\alpha,\beta) \neq 0$

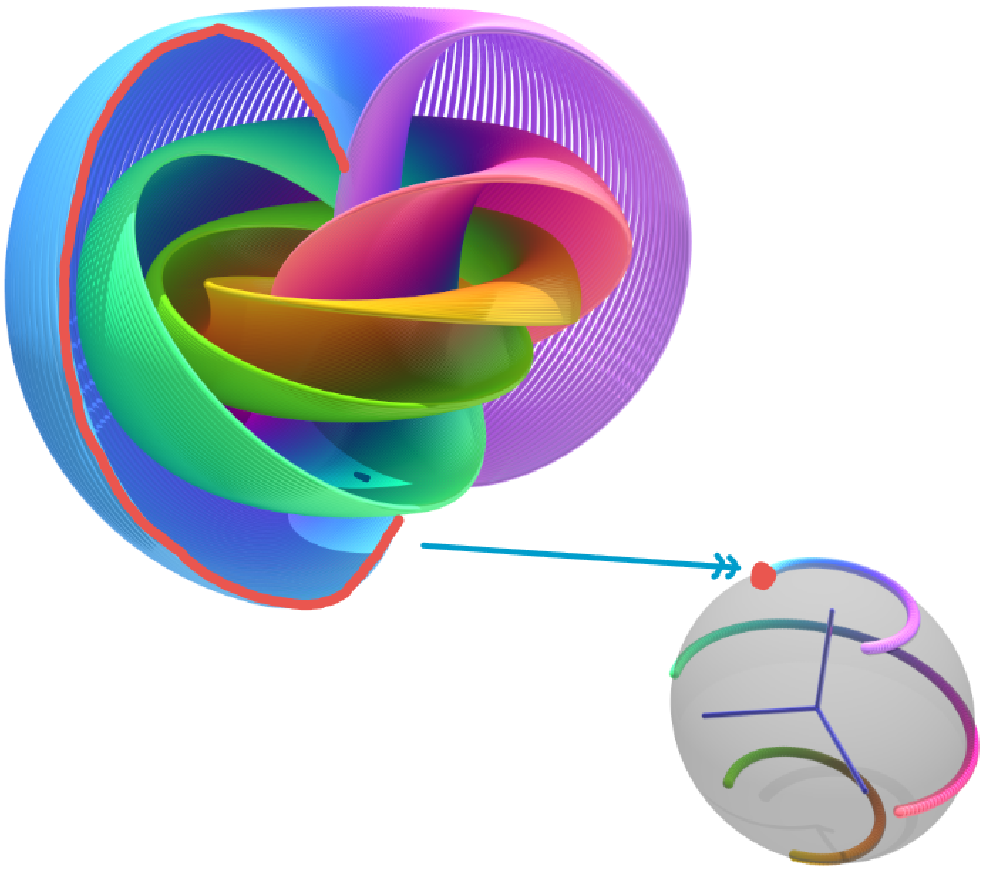
$$Y := S^3 \setminus \text{two unlinked circles}$$

identical cohomology but $\mu_2^Y(\alpha,\beta) = 0$

The Hopf fibration

$$\mathbb{C}^2 \supseteq S^3 \twoheadrightarrow S^2 \subseteq \mathbb{C} \times \mathbb{R}$$

$$(w,z) \mapsto (2w\bar{z}, |w|^2 - |z|^2)$$



A_∞-structure on singular cohomology

X : top. space

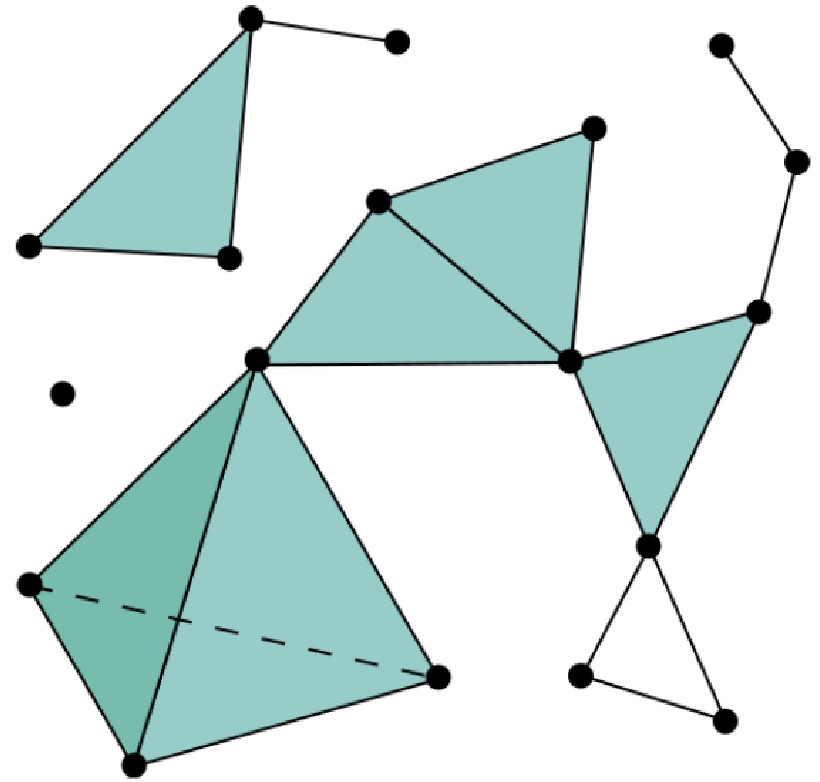
$$H^*(X;\mathbb{R}) := \bigoplus_{p=0}^{\infty} H^p(X;\mathbb{R})$$

admits operations

$$d := \mu_1, \mu_2, \mu_3, \ldots$$

that satisfy the A_∞-equations, where

$d = 0$ and μ_2 : cup product



What is do the higher operations tell us about the space X ?

Massey's beautiful theorem

$X := S^3 \setminus$ Borromean rings

$H^0(X;\mathbb{R}) = \mathbb{R}$

$H^1(X;\mathbb{R}) = \mathbb{R}\alpha \oplus \mathbb{R}\beta \oplus \mathbb{R}\gamma$

$H^2(X;\mathbb{R}) = \mathbb{R}$

$$\mu_2^X(\alpha,\beta) = \mu_2^X(\alpha,\gamma) = \mu_2^X(\beta,\gamma) = 0$$

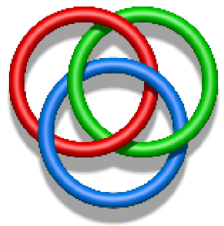
The same is true for

$Y := S^3 \setminus$ three unlinked circles

Thm (Massey 1969)

For $X := S^3 \setminus$ Borromean rings

$$\mu_3^X(\alpha,\beta,\gamma) \neq 0$$



but for

$Y := S^3 \setminus$ three unlinked circles

$$\mu^Y(\alpha,\beta,\gamma) = 0$$

03

AN APPLICATION TO ALGEBRAIC GEOMETRY

The Donovan-Wemyss Conjecture

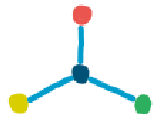
$$\boxed{\text{3D compound Du Val singularities + cond.}} \xrightarrow{\text{Donovan-Wemyss}} \boxed{\text{Finite-dim. algebras}}$$

$p : X \longrightarrow \mathrm{Spec}(R)$

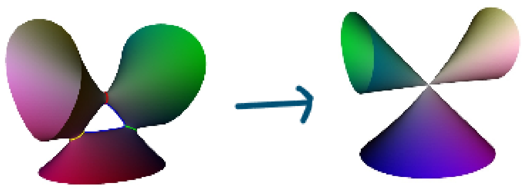
X: crepant res.

$\mathrm{Spec}(R)$: cDV sing

$\Lambda(R,p)$: contraction algebra



$\mathbb{D}_4$



2 dim analogue

Conj (Donovan-Wemyss 2013)

$$\Lambda(R_1,p_1) \overset{\text{der}}{\sim} \Lambda(R_2,p_2) \iff R_1 \cong R_2$$

Wemyss 2018, Hua-Keller 2024, August 2020

Algebraic Problem $\Longrightarrow$ DW Conjecture

Solved in 2022 using A_∞-structures in joint work with F. Muro

2024: New proof by Karmazyn-Lepri-Wemyss also using A_∞-structures!