On length categories and their taxonomy

Def (Grothendieck 1957)

An additive cat $\mathcal{A}$ is abelian if

$\forall f : X \to Y$ in $\mathcal{A}$ $\exists \ker f \to X$ $\exists Y \to \operatorname{coker} f$ and

$$\begin{tikzcd} \ker f \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] & \operatorname{coker} f \\ & \operatorname{coim} f \arrow[r, "\sim"] \arrow[r, phantom, shift left=4, "="] & \operatorname{im} f \arrow[u] & \end{tikzcd}$$

Ex R: ring $\rightsquigarrow$ $\operatorname{Mod}(R)$: cat of (right) R-modules

Ex X: top. space $\rightsquigarrow$ $\operatorname{Shv}(X)$: cat of sheaves of abelian groups on X

Gabriel 1973 : Indecomposable Representations II

Def $0 \neq S \in \mathcal{A}$ is simple if $\forall X \hookrightarrow S$ mono $(X = 0$ or $S/X = 0)$

Def (Gabriel 1962, 1973)

An abelian cat. $\mathcal{A}$ is a length cat. if it is ess. small and

$\forall X \in \mathcal{A}$ $\exists$ $0 = X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_\ell = X$ s.t.

$\forall 1 \leq i \leq \ell$ X_i / X_{i-1} is simple.

Ex R: ring $\rightsquigarrow$ $\operatorname{fl}(R)$: cat of finite-length R-modules

Ex Bernstein-Gelfand-Gelfand cat. $\mathcal{O}$ (1976)

Def $\mathcal{A}$: abelian cat.

- $S \in \mathcal{A}$ is a <u>brick</u> if $\mathrm{End}_{\mathcal{A}}(S)$ is a div. ring.
- S, T are <u>orthogonal</u> if $\mathrm{Hom}_{\mathcal{A}}(S,T) = 0$

Thm (Ringel 1976) $\mathcal{S} \subseteq \mathcal{A}$: set of pairwise orthogonal bricks

$\mathrm{Filt}(\mathcal{S}) = \{ X \in \mathcal{S} \mid X \text{ admits an } \mathcal{S}\text{-filtration} \} \subseteq \mathcal{A}$

is an exact length subcat with simple objects $S \in \mathcal{S}$.

Def $\mathcal{A}$: length cat.

- The <u>Loewy length</u> of $X \in \mathcal{A}$ is the smallest $\ell \geq 0$ such that

  $\exists\, 0 = X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_\ell = X$ s.t.
  $\forall\, 1 \leq i \leq \ell \quad X_i / X_{i-1}$ is <u>semi</u>-simple

- The <u>height</u> of $\mathcal{A}$ is $\sup \{ \text{Loewy Length}(X) \mid X \in \mathcal{A} \}$

Rmk The objects of Loewy length $N \geq 0$ form a length subcategory $\mathcal{A}_N \subseteq \mathcal{A}$ (need not be extension-closed)

Thm (Gabriel 1962, 1973) $\mathcal{A}$: length cat. TFAE

(1) $\exists\, R$: right artinian st. $\mathcal{A} \simeq \mathrm{fl}(R)$

(2) TFSH:

- The height of $\mathcal{A}$ is finite
- There are only finitely many simple objects in $\mathcal{A}$ up to iso
- $\forall\, S, T \in \mathcal{A}$: simple $\mathrm{Ext}^1_{\mathcal{A}}(S,T)$ is finite length over $\mathrm{End}(T)$

Def (Gabriel 1962, 1973)

- A complete Hausdorff top. ring R is <u>pseudocompact</u> if $\forall\, 0 \in U \subseteq R$: open $\exists\, I \underset{\text{right ideal}}{\overset{\text{open}}{\subseteq}} R$ s.t. $I \subseteq U$ & R/I has finite length
  R is <u>basic</u> iff $R/\mathrm{rad}\,R$ is a product of division rings.

- $\mathrm{disc}(R)$: cat. of finite-length <u>discrete</u> R-modules ($\mathrm{ann}\, M \subseteq R$: open)

Ex $R = \mathbb{C}[[X]]$ is a pseudocompact ring (J-adic topology)

Thm (Gabriel 1962) $\mathcal{A}$: abelian cat TFAE

(1) $\exists !$ R : basic pseudocompact ring s.t. $\mathcal{A} \simeq \mathrm{disc}(R)$
(2) $\mathcal{A}$ is a length cat.

Gabriel in SGA3 VII_B (1962-1964)

Thm (Kiełpiński - Simson - Tyc 1973, Witkowski 1979) k : field

$$(\mathrm{PCAlg}_k)^{op} \xrightarrow{\sim} \mathrm{Coalg}_k, \quad R \mapsto R^\circ := \underbrace{\mathrm{hom}_k}_{\text{continuous hom}}(R, k),$$

$\mathrm{disc}(R) \cong R^\circ\text{-comod}$: cat of fin.-dim. R°-comodules

Thm (Takeuchi 1977) $\mathcal{A}$: abelian k-cat, k : field TFAE

(1) $\exists\, C$: coalgebra st. $\mathcal{A} \simeq C$-comod

(2) $\mathcal{A}$ is a length cat.

(Kleiner - Roiter 1977, Roiter 1979, Drozd 1979, ...,
Crawley-Boevey 1988, Burt - Butler 1991, ...
König - Külshammer - Ovsienko 2014, ...) BOCS $\sim$ $\overset{\text{exact}}{\text{length}}$ cat's

Def (Gabriel 1973) $\mathcal{A}$: length cat.

- S_i, $i \in I$: complete set of representatives of the isoclasses of simple objects in $\mathcal{A}$

- $K_i := \mathrm{End}_{\mathcal{A}}(S_i)$: division ring

- ${}_jE_i := \mathrm{Ext}^1_{\mathcal{A}}(S_i, S_j)$: K_j-K_i-bimodule

- We say that $\mathcal{A}$ is of species $(*) = \left((K_i)_{i \in I}, ({}_jE_i)_{i,j \in I} \right)$

- A species is abstract data $(*)$ as above.

- A k-species, k a field, is a species s.t. $\forall i \in I$ K_i is a k-algebra

- A k-quiver is a k-species s.t. $\forall i \in I$ $K_i = k$.

$$\leftrightsquigarrow \text{ quiver with } \begin{cases} \text{vertex set } I \\ \#(i \to j) = \dim_k({}_jE_i) \end{cases}$$

Ex $\mathrm{fl}(\mathbb{C}[[X]])$ and $\mathrm{fl}(\mathbb{C}[X]/(X^N))$, $N \geq 2$, are of the same species : $\bullet \circlearrowleft$

Ex $\mathrm{fl}\begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$ is of species $\mathbb{R} \xleftarrow{\mathbb{C}} \mathbb{C}$ (Dynkin type B_2)

Def $\mathcal{A}$: length cat. The global dimension of $\mathcal{A}$ is

$$\mathrm{gldim}\, \mathcal{A} := \inf\{ n \geq 0 \mid \forall i > d \ \mathrm{Ext}^i_{\mathcal{A}} = 0 \}$$

Ex $\mathcal{A}$ is semisimple $\Leftrightarrow$ gldim $\mathcal{A} = 0$

In this case $\mathcal{A} \simeq \prod_{i \in I} \mathrm{fl}(K_i)$, K_i : division ring

Def $\mathcal{A}$ is <u>hereditary</u> if gldim $\mathcal{A} \leq 1$

Construction (Gabriel 1973) $((K_i)_{i\in I}, ({}_jE_i)_{i,j\in I})$: species

$R := \hat{T}_K(\omega)$: <u>completed tensor K-algebra</u> of ω where

$$K := \prod_{i\in I} K_i, \quad {}_i\omega_j := \mathrm{hom}_{K_j}({}_jE_i, K_j), \quad \omega := \prod_{i,j\in I} {}_i\omega_j,$$

Then disc(R) is a length category of species
$((K_i)_{i\in I}, ({}_jE_i)_{i,j\in I})$

Rmk For a k-quiver, $\hat{T}_K(\omega)$ is the completed path k-alg.

Thm (Gabriel 1973, 1980)

A : fin.-dim alg / $k = \bar{k}$

$\Rightarrow$ A is Morita equivalent to kQ_A / I, $I \subseteq J^2$ (k-quiver of A)

J : arrow ideal

Rmk If $k = \bar{k}$, the above thm reduces the rep. theory of finite-dim. algebras to the rep. theory of quivers with <u>admissible</u> relations.

Prop (Folklore)

The Krull-Remak-Schmidt Thm holds in a length cat:

$\forall X \in \mathcal{A} \ \exists X \cong X_1 \oplus \cdots \oplus X_n$, $\forall i$: $\mathrm{End}(X_i)$: local ring
unique up to iso and permutation of the direct summands

Problem Given a length cat. $\mathcal{A}$, classify its indecomposable obj's.

Warning $\mathrm{fl}(\mathbb{C}\langle x,y\rangle)$ is intractable (wild)

Def $\mathcal{A}$ is of finite type if there are only finitely many indecomposable objects in $\mathcal{A}$ up to isomorphism.

Def An Artin algebra R is an algebra over a comm. artinian ring k st. R has finite length as k-module.

Thm (Auslander 1971)

There is a 1-1 correspondence between:

(1) Morita eq. classes of Artin alg's. R st.

$$\mathrm{fl}(R) \text{ is of finite type}$$

(2) ——————— " ——————— Γ st.

$$\mathrm{gl\,dim}\,\Gamma \leq 2 \leq \mathrm{dom.dim}\,\Gamma$$

(1) $\to$ (2) is given by $R \mapsto \mathrm{End}_R(M)$, $\mathrm{add}(M) = \mathrm{fl}(R)$.

(Auslander - Reiten 1975)

Almost-split sequences $\rightsquigarrow$ AR theory

"Projective resolutions of simple Π-modules of proj. dim 2"

(Iyama 2007)

Replacing "2" by "d+1" leads to the discovery of d-cluster tilting modules and higher AR theory

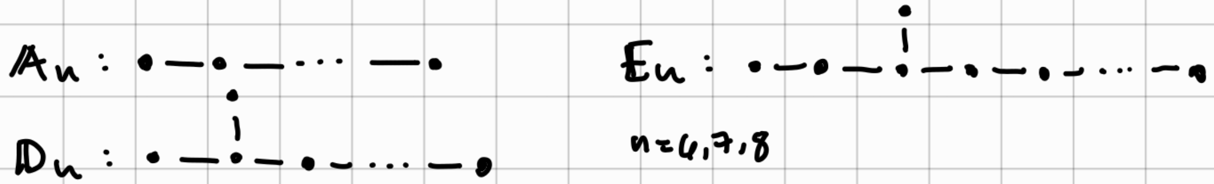
"Projective resolutions of simple Π-modules of proj. dim $d+1$"

<u>Thm</u> (Yoshii 1956, Bäckström, Gabriel, Kleiner 1972)

$\Bbbk$: field, Q : connected acyclic quiver. TFAE

(1) The path algebra $\Bbbk Q$ is of finite representation type.

(2) The underlying graph of Q is a Dynkin diagram of type $\mathbb{A}$, $\mathbb{D}$, $\mathbb{E}_n$, $n = 6, 7, 8$



(Tits ?)

In this case, the number of indec. rep's of $\Bbbk Q$ equals the number of positive roots of the corresponding root system.

(Bernstein - Gelfand - Ponomarev 1973)

Proof using the concept of <u>reflection functors</u>.

(Dlab-Ringel 1976)

Extension to k-species. All Dynkin types <u>can</u> appear, and do appear for suitable fields.

(Dowbor-Ringel-Simson 1980)

Extension to hereditary artinian rings. All finite Coxeter types <u>can</u> appear. $I_2(5)$ can be realised (Schofield)

$\boxed{I_2(p),\ p \geq 7 \text{ seems to be open}}$

- Kac's Thm (1980)

- Ringel-Hall algebras (1990)

- Fomin-Zelevinsky cluster algebras (2002)

  - Additive categorifications:
    - Buan-Marsh-Reiten-Reineke-Todorov (2006)
    - Derksen-Weyman-Zelevinsky (2008, 2010)
    - Geiß-Leclerc-Schröer (2006-2018)
    - Demonet (2010)
    - Labardini-Fragoso-Zelevinsky (2016)

(*) Variants of the concept of species:

(GLS) species over truncated polynomial algebras
(Dem) species over group algebras
(LF-Z) species with potential

- Pro-species of algebras (Külshammer 2017)

Prop (Beilinson-Bernstein-Deligne-Gabber 1982)

$\mathcal{D}^b(\mathcal{A})$ admits a bounded t-structure with heart $\mathcal{A}$

Rmk $\mathcal{S} = \{S_i \mid i \in I\} \subseteq \mathcal{A}$: simples of $\mathcal{A}$.

$\rightsquigarrow \operatorname{thick}(\mathcal{S}) = \mathcal{D}^b(\mathcal{A})$

(Keller 1994) $\operatorname{perf}(\mathcal{S}_{dg}) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{A})$

Thm (Al-Nofayee 2009, Schnürer 2011, Keller-Nicolás 2013, Keller-Yang 2014, Rickard-Rouquier 2017, Su-Yang 2019)

$\mathcal{T}$: ess. small tri. cat.

$\cup$

$\mathcal{S} = \{S_i \mid i \in \overset{\text{set}}{I}\}$ pairwise non-isomorphic s.t.

- $\forall i,j \in I \quad \mathcal{T}(S_i, S_j) = \begin{cases} \text{division ring} & i = j \\ 0 & i \neq j \end{cases}$

- $\forall i,j \in I \quad \mathcal{T}(S_i, \Sigma^{<0}(S_j)) = 0$

$\Longrightarrow$

$\operatorname{tri}(\mathcal{S})$ admits a bounded t-structure with length heart

Rmk Bounded t-structures with length heart play an important role in the study of Bridgeland's stability manifold.

Ex $D^{fd}(A)$, A: proper connective dga, $\mathcal{S}$: simple $H^0(A)$-mod's