(Non-)uniqueness of strong enhancements

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Conventions

$\mathbf{k}$: perfect field, *e.g.* $\operatorname{char}(\mathbf{k}) = 0$ or $\mathbf{k} = \overline{\mathbf{k}}$

Λ : finite-dimensional $\mathbf{k}$-algebra

$\mathcal{P}(\Lambda)$: finitely-generated projective (right) Λ-modules

DG enhancements

Why the need for DG enhancements?

Abelian World	Triangulated World
$\mathcal{A},\ \mathcal{B}$: abelian cats.	$\mathcal{S},\ \mathcal{T}$: triangulated cats.
$\mathsf{Fun}(D,\mathcal{B})$: abelian cat.	$\mathsf{Fun}(D,\mathcal{T})$: not triangulated
$\exists\ \mathcal{A}\otimes\mathcal{B}$ e.g. in Grothendieck case	$\nexists\ \mathcal{S}\otimes\mathcal{T}$ even in well-gen. case
$\exists$ 2-pullback along exact functors	$\nexists$ 2-pullback along exact functors
$\vdots$	$\vdots$

DG World: Vector spaces $\rightsquigarrow$ Complexes of vector spaces

The passage to the DG world solves 'all of our problems'… and introduces new ones!

Reminder: Keller's Theorem

$\mathcal{T}$: algebraic triangulated category

Keller (1994): The following statements hold:

- $\mathcal{T}$: ess. small & idempotent-complete

$$\exists G \in \mathcal{T},\ \mathsf{thick}(G) = \mathcal{T} \qquad \Longleftrightarrow \qquad \exists A\colon \text{DG algebra},\ \mathrm{D}^{c}(A) \simeq \mathcal{T}$$

- $\mathcal{T}$: with small coproducts

$$\exists G \in \mathcal{T}\colon \text{compact},\ \mathsf{Loc}(G) = \mathcal{T} \qquad \Longleftrightarrow \qquad \exists A\colon \text{DG algebra},\ \mathrm{D}(A) \simeq \mathcal{T}$$

All triangulated cats in representation theory are controlled by DG algebras / DG cats.

Pre-triangulated DG categories

Bondal–Kapranov (1990) $\mathcal{A}$: ess. small DG category is **(Karoubian) pre-triangulated** if

$$\mathbf{y}\colon \mathcal{A} \hookrightarrow \mathsf{D}^c(\mathcal{A})_{\mathsf{dg}}, \qquad a \longmapsto \mathcal{A}(-,a),$$

induces an equivalence

$$H^0(\mathbf{y})\colon H^0(\mathcal{A}) \overset{\sim}{\hookrightarrow} \mathsf{D}^c(\mathcal{A})$$

Remark: $H^0(\mathcal{A})$ is then a triangulated category

(Keller's Theorem + Drinfeld–Verdier quotient) $\implies$ All triangulated categories in representation theory admit a DG enhancement (see next)

Enhancements of triangulated categories

Bondal–Kapranov (1990): DG enhancement $\mathcal{A}$ of $(\mathcal{T}, \Sigma, \triangle)$

- $\mathcal{A}$: pre-triangulated DG category
- $\exists \Phi \colon H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{T}$: equivalence of triangulated categories

$\mathcal{A} \sim \mathcal{B}$ generated by

$$\exists f \colon \mathcal{A} \xrightarrow{\text{quasi-equiv.}} \mathcal{B}$$

$$H^0(f) \colon H^0(\mathcal{A}) \xrightarrow{\sim} H^0(\mathcal{B})$$

$\mathrm{DGE}_3(\mathcal{T}, \Sigma, \triangle)$

Equivalence classes of DG enhancements

$(\mathcal{T}, \Sigma, \triangle)$ admits a **unique DG enhancement** if $\mathrm{DGE}_3(\mathcal{T}, \Sigma, \triangle) = \{*\}$

(Non-)uniqueness of DG enhancements

The following (**k**-linear) triangulated categories admit **unique DG enhancements**:

Keller (1994), Lunts–Orlov (2010), Canonaco–Stellari (2018), Canonaco–Neeman–Stellari (2022): All derived & homotopy categories of abelian categories, 'algebro-geometric' derived categories…

Muro (2022) $d = 1$, **J–Muro (2022)** $d \geq 1$: Hom-finite, Krull–Schmidt, algebraic triangulated categories with a $d\mathbb{Z}$-cluster tilting object

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Rizzardo-Van den Bergh (2019, 2020): **k**-linear triangulated categories with **non-unique DG enhancements** and **without any DG enhancements** at all

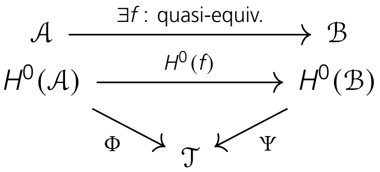
Schlichting (2002), Dugger–Shipley (2007): $\mathbb{Z}$-linear algebraic triangulated categories with **non-unique DG enhancements**

Strong enhancements of triangulated categories

Lunts–Orlov (2010) Strong DG enhancement $(\mathcal{A}, \Phi)$ **of** $(\mathcal{T}, \Sigma, \vartriangle)$

- $\mathcal{A}$: pre-triangulated DG cat
- $\Phi \colon H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{T}$: equivalence of triangulated categories

$(\mathcal{A}, \Phi) \sim (\mathcal{B}, \Psi)$ generated by



$\mathrm{SDGE}_3(\mathcal{T}, \Sigma, \vartriangle)$

Equivalence classes of strong DG enhancements

$(\mathcal{T}, \Sigma, \vartriangle)$ admits a **unique strong DG enhancement** if $\mathrm{SDGE}_3(\mathcal{T}, \Sigma, \vartriangle) = \{*\}$

Uniqueness of strong enhancements

Lunts–Orlov (2010), Canonaco–Stellari (2017), Olander (2020, 2022), Li–Pertusi–Zhao (2022): Uniqueness of strong DG enhancements for various algebraic triangulated categories of 'algebro-geometric origin'

Chen–Ye (2018), Lorenzin (2022) Bounded derived cats. of hereditary abelian cats.

Question: Does there exist an algebraic triangulated category with a unique DG enhancement but non-unique strong DG enhancements?

Yes!

… otherwise why this talk …

DG enhancements in HHA

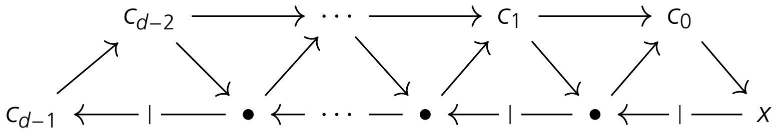
$d\mathbb{Z}$-cluster tilting subcategories ($d \geq 1$)

Iyama–Yoshino (2008), Geiß–Keller–Oppermann (2013), Beligiannis (2015)

$\mathcal{T}$: Hom-finite + Krull–Schmidt triangulated category & $\mathcal{C} = \mathrm{add}(c)$

$\mathcal{C} \subseteq \mathcal{T}$: ***d*-cluster-tilting** if $\forall 0 < i < d$, $\mathcal{T}(\mathcal{C}, \mathcal{C}[i]) = 0$ and $\mathcal{T} = \mathcal{C} * \mathcal{C}[1] * \cdots * \mathcal{C}[d-1]$

$\forall x \in \mathcal{T}$ $\exists$

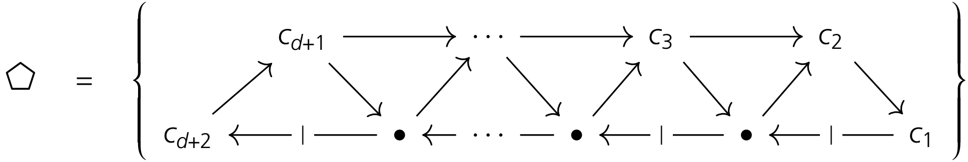


with $c_0, c_1, \ldots, c_{d-1} \in \mathcal{C}$.

Standard (d+2)-angulated categories

$\mathcal{T}$: triangulated category

$\mathcal{C} \subseteq \mathcal{T}$: $d\mathbb{Z}$-cluster tilting subcategory = d-cluster tilting + $\mathcal{C} = \Sigma^d(\mathcal{C})$



Geiß–Keller–Oppermann (2013)

The triple $(\mathcal{C}, \Sigma^d, \pentagon)$ is a $(d+2)$-angulated category

Twisted (d+2)-periodic algebras

$(\mathcal{F}, \Sigma, \pentagon)$: $(d+2)$-angulated category + Hom-finite + Krull–Schmidt

Suppose $\exists c \in \mathcal{F}$ basic object s.t. $\operatorname{add}(c) = \mathcal{F} \quad \rightsquigarrow \quad \Lambda := \mathcal{F}(c, c)$

Freyd (1966) + Heller (1968) $d = 1$, Geiss–Keller–Opermann (2013), Green–Snashal–Solberg (2003), Hanihara (2022)

- Λ : basic Frobenius algebra
- $\exists\ \sigma : \Lambda \xrightarrow{\sim} \Lambda$ algebra automorphism s.t.

$$\Omega_{\Lambda}^{d+2} \overset{!}{\cong} (-)_{\sigma} : \underline{\mathrm{mod}}(\Lambda) \xrightarrow{\sim} \underline{\mathrm{mod}}(\Lambda)$$

- $\Omega_{\Lambda^e}^{d+2}(\Lambda) \simeq \Lambda_{\sigma}$ in $\underline{\mathrm{mod}}(\Lambda^e)$

Amiot–Lin (d+2)-angulations

Λ : basic Frobenius algebra & $\sigma\colon\Lambda\xrightarrow{\sim}\Lambda$

$\Sigma := -\otimes_\Lambda \Lambda_{\sigma^{-1}} : \mathcal{P}(\Lambda)\xrightarrow{\sim}\mathcal{P}(\Lambda)$

$\eth\in\operatorname{Ext}^{d+2}_{\Lambda^e}(\Lambda,\Lambda_\sigma) : 0\to\Lambda_\sigma\to P_{d+1}\to\cdots\to P_1\to P_0\to\Lambda\to 0,\quad P_i\in\mathcal{P}(\Lambda^e)$

$\pentagon_\eth = \{Q_{d+2}\to Q_{d+1}\to\cdots\to Q_1\to\Sigma Q_{d+2}\}$

'Exact sequences in $\mathcal{P}(\Lambda)$ satisfying certain exactness conditions rel $\eth$'

Amiot (2008) $d=1$**, Lin (2019)**

The triple $(\mathcal{P}(\Lambda),\Sigma,\pentagon_\eth)$ is a $(d+2)$-angulated category

J–Muro (2022) Up to equivalence, $\pentagon_\eth$ independent of the choice of $\eth$

Pre-(d+2)-angulated DG categories

$\mathcal{A}$: DG category is **(Karoubian) pre-(d+2)-angulated** if

$$\mathsf{y}\colon \mathcal{A} \hookrightarrow \mathrm{D}^{\mathrm{c}}(\mathcal{A})_{\mathrm{dg}}, \qquad a \longmapsto \mathcal{A}(-,a),$$

induces an equivalence

$$H^0(\mathsf{y})\colon H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{C} \subseteq \mathrm{D}^{\mathrm{c}}(\mathcal{A})$$

with a $d\mathbb{Z}$-cluster tilting subcategory of $\mathcal{C} \subseteq \mathrm{D}^{\mathrm{c}}(\mathcal{A})$

Remark: $H^0(\mathcal{A})$ is then a standard $(d+2)$-angulated category

Remark: (Karoubian) pre-$(1+2)$-angulated = (Karoubian) pre-triangulated

Enhancements of (d+2)-angulated categories

DG enhancement $\mathcal{A}$ of $(\mathcal{F}, \Sigma, \pentagon)$

- $\mathcal{A}$: pre-$(d+2)$-angulated DG category
- $\exists\Phi\colon H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{F}$: equivalence of $(d+2)$-angulated categories

$\mathcal{A} \sim \mathcal{B}$ generated by

$$\mathcal{A} \xrightarrow{\exists f\,:\,\text{quasi-eq}} \mathcal{B}$$

$$H^0(\mathcal{A}) \xrightarrow{H^0(f)} H^0(\mathcal{B})$$

$\mathrm{DGE}_{d+2}(\mathcal{F}, \Sigma, \pentagon)$

Equivalence classes of DG enhancements

$(\mathcal{F}, \Sigma, \pentagon)$ admits a **unique DG enhancement** if $\mathrm{DGE}_{d+2}(\mathcal{F}, \Sigma, \pentagon) = \{*\}$

Uniqueness of pre-(d+2)-angulated DG enhancements

Λ : basic Frobenius algebra & $\sigma\colon\Lambda\xrightarrow{\sim}\Lambda$ s.t. $\Omega^{d+2}_{\Lambda^e}(\Lambda)\simeq\Lambda_\sigma$

$\Sigma := -\otimes_\Lambda \Lambda_{\sigma^{-1}}\colon \mathcal{P}(\Lambda)\xrightarrow{\sim}\mathcal{P}(\Lambda)$

$\pentagon$: Amiot–Lin $(d+2)$-angulation of $(\mathcal{P}(\Lambda),\Sigma)$

Muro (2022) $d=1$, **J–Muro (2022)** $d\geq 1$:

- $(\mathcal{P}(\Lambda),\Sigma,\pentagon)$ admits a DG enhancement and it is moreover unique.
- Up to equivalence,

$$\exists!\ \mathcal{T}\colon \textbf{algebraic}\text{ triangulated. cat.,}\qquad (\mathcal{P}(\Lambda),\Sigma,\pentagon)\xrightarrow{\simeq}\mathcal{C}\subseteq\mathcal{T},$$

where $\mathcal{C}\subseteq\mathcal{T}$ is a $d\mathbb{Z}$-cluster tilting subcategory. Moreover, $\mathcal{T}$ admits a **unique DG enhancement**.

Strong DG enhancements

Strong enhancements of (d+2)-angulated categories

Strong DG enhancement $(\mathcal{A},\Phi)$ **of** $(\mathcal{F},\Sigma,\pentagon)$

- $\mathcal{A}$: pre-$(d+2)$-angulated DG cat
- $\Phi\colon H^0(\mathcal{A})\xrightarrow{\sim}\mathcal{F}$: equivalence of $(d+2)$-angulated categories

$(\mathcal{A},\Phi)\sim(\mathcal{B},\Psi)$ generated by

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\exists f\,:\ \text{quasi-eq}} & \mathcal{B} \\ H^0(\mathcal{A}) & \xrightarrow{H^0(f)} & H^0(\mathcal{B}) \\ & \searrow^{\Phi} \quad \mathcal{F} \quad {}^{\Psi}\swarrow & \end{array}$$

$\mathrm{SDGE}_{d+2}(\mathcal{F},\Sigma,\pentagon)$

Equivalence classes of strong DG enhancements

$(\mathcal{F},\Sigma,\pentagon)$ admits a **unique strong DG enhancement** if $\mathrm{SDGE}_{d+2}(\mathcal{F},\Sigma,\pentagon)=\{*\}$

Pre-triangulated vs pre-(d+2)-angulated enhancements

$\mathcal{T}$: algebraic triangulated category + Hom-finte + Krull–Schmidt

$c \in \mathcal{T}$: $d\mathbb{Z}$-cluster tilting object $\rightsquigarrow \mathcal{C} := \operatorname{add}(c) \subseteq \mathcal{T}$

There is a canonical **restriction map**

$$\mathrm{SDGE}_3(\mathcal{T}, \Sigma, \triangle) \longrightarrow \mathrm{SDGE}_{d+2}(\mathcal{C}, \Sigma^d, \pentagon)$$

$$[\mathcal{A}] \longmapsto [\mathcal{A}_{\mathcal{C}}]$$

$$\begin{array}{ccc} H^0(\mathcal{A}) & \hookleftarrow & H^0(\mathcal{A}_{\mathcal{C}}) \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{T} & \hookleftarrow & \mathcal{C} \end{array}$$

Open question: Is this map is injective or surjective if $d \geq 2$?

Main results

The stable centre and the map $\zeta^{\times}$

Λ : basic Frobenius algebra

$$\begin{array}{ccccccc}
[\Lambda^e](\Lambda,\Lambda) & \xleftrightarrow{\sim} & \mathrm{U}(\Lambda) & & & & \\
\downarrow & & \downarrow & & & & \\
\mathrm{Hom}_{\Lambda^e}(\Lambda,\Lambda) & \xleftrightarrow{\sim} & \mathrm{Z}(\Lambda) & \xrightarrow{\sim} & \mathrm{Z}(\mathrm{mod}(\Lambda)) & = & \mathrm{End}\left(1_{\mathrm{mod}(\Lambda)}\right) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\underline{\mathrm{Hom}}_{\Lambda^e}(\Lambda,\Lambda) & \xleftrightarrow{\sim} & \underline{\mathrm{Z}}(\Lambda) & \xrightarrow{\zeta} & \mathrm{Z}(\underline{\mathrm{mod}}(\Lambda)) & = & \mathrm{End}\left(1_{\underline{\mathrm{mod}}(\Lambda)}\right) \\
\| & & \uparrow & & \uparrow & & \| \\
\underline{\mathrm{HH}}^0(\Lambda) & & \underline{\mathrm{Z}}(\Lambda)^{\times} & \xrightarrow{\zeta^{\times}} & \mathrm{Z}(\underline{\mathrm{mod}}(\Lambda))^{\times} & & \mathrm{HH}^0(\underline{\mathrm{mod}}(\Lambda)) \\
 & & \| & & \| & & \\
 & & \underline{\mathrm{HH}}^0(\Lambda)^{\times} & & \mathrm{HH}^0(\underline{\mathrm{mod}}(\Lambda))^{\times} & &
\end{array}$$

Main theorem (J–Muro 2022)

Λ : basic Frobenius algebra & $\sigma\colon \Lambda \xrightarrow{\sim} \Lambda$ s.t. $\Omega^{d+2}_{\Lambda^e}(\Lambda) \simeq \Lambda_\sigma$

$\Sigma := -\otimes_\Lambda \Lambda_{\sigma^{-1}} \colon \mathcal{P}(\Lambda) \xrightarrow{\sim} \mathcal{P}(\Lambda)$

$\pentagon$: Amiot–Lin $(d+2)$-angulation of $(\mathcal{P}(\Lambda), \Sigma)$

There are bijections:

$$\begin{array}{ccccc}
\mathrm{SDGE}_{d+2}(\mathcal{P}(\Lambda),\Sigma,\pentagon) & \hookrightarrow & \mathrm{SDGE}_{d+2}(\mathcal{P}(\Lambda),\Sigma) & & \\
\updownarrow\wr & & \updownarrow\wr & & \\
\ker \zeta^{\times} & \hookrightarrow & \underline{Z}(\Lambda)^{\times} & \xrightarrow{\zeta^{\times}} & Z(\underline{\mathrm{mod}}(\Lambda))^{\times}
\end{array}$$

$(\mathcal{P}(\Lambda), \Sigma, \pentagon)$ admits a unique strong DG enhancement $\iff$ $\ker \zeta^{\times} = 1$

The case d=1 – A complete answer

Λ : basic Frobenius algebra & $\sigma\colon \Lambda \xrightarrow{\sim} \Lambda$ s.t. $\Omega^3_{\Lambda^e}(\Lambda) \simeq \Lambda_\sigma$

$\Sigma := - \otimes_\Lambda \Lambda_{\sigma^{-1}} : \mathcal{P}(\Lambda) \xrightarrow{\sim} \mathcal{P}(\Lambda)$

$\pentagon$: Amiot triangulation of $(\mathcal{P}(\Lambda), \Sigma)$

There are bijections:

$$\begin{array}{ccccc}
\mathrm{SDGE}_3(\mathcal{P}(\Lambda), \Sigma, \triangle) & \hookrightarrow & \mathrm{SDGE}_3(\mathcal{P}(\Lambda), \Sigma) & & \\
\updownarrow\wr & & \updownarrow\wr & & \\
\ker \zeta^\times & \hookrightarrow & \underline{Z}(\Lambda)^\times & \xrightarrow{\zeta^\times} & Z(\underline{\mathrm{mod}}(\Lambda))^\times
\end{array}$$

$(\mathcal{P}(\Lambda), \Sigma, \triangle)$ admits a unique strong DG enhancement $\iff$ $\ker \zeta^\times = 1$

The algebra of dual numbers – An explicit example

$\Lambda = \mathbf{k}[\varepsilon]$: algebra of dual numbers

$\Sigma = - \otimes_{\Lambda}\, {}_1\Lambda_{\sigma^{-1}} : \mathcal{P}(\Lambda) \xrightarrow{\sim} \mathcal{P}(\Lambda), \qquad \sigma : \varepsilon \longmapsto -\varepsilon$

$\triangle$: Amiot triangulation of $(\mathcal{P}(\Lambda), \Sigma)$

$$
\begin{array}{ccc}
Z(\Lambda) & \xleftrightarrow{\sim} & Z(\mathrm{mod}(\Lambda)) \\
\twoheaddownarrow & & \downarrow \\
\underline{Z}(\Lambda) & \longrightarrow & Z(\underline{\mathrm{mod}}(\Lambda)) \\
\hookuparrow & & \hookuparrow \\
\underline{Z}(\Lambda)^{\times} & \xrightarrow{\zeta^{\times}} & Z(\underline{\mathrm{mod}}(\Lambda))^{\times}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Lambda & \xleftrightarrow{\sim} & \Lambda \\
\twoheaddownarrow & & \downarrow \\
\Lambda/(2\varepsilon) & \xrightarrow{\zeta} & \mathbf{k} \\
\hookuparrow & & \hookuparrow \\
\underline{Z}(\Lambda)^{\times} & \xrightarrow{\zeta^{\times}} & \mathbf{k}^{\times}
\end{array}
$$

$$\mathrm{char}(\mathbf{k}) \neq 2 \quad \Longrightarrow \quad \mathrm{SDGE}_3(\mathcal{P}(\Lambda), \Sigma, \triangle) = \ker \zeta^{\times} = 1$$

$$\mathrm{char}(\mathbf{k}) = 2 \quad \Longrightarrow \quad \mathrm{SDGE}_3(\mathcal{P}(\Lambda), \Sigma, \triangle) = \ker \zeta^{\times} = 1 + (\varepsilon)$$

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