(Non-)uniqueness of strong enhancements

GUSTAVO JASSO (Centre for Mathematical Sciences)

JOINT WORK WITH FERNANDO MURO (Universidad de Sevilla)

Motivation

Theorem (Keller 1994)

$\mathcal{T}$: ess. small + idempotent complete + algebraic triangulated category

$$\exists G \in \mathcal{T},\ \operatorname{thick}(G) = \mathcal{T} \qquad \Longleftrightarrow \qquad \exists A\text{: DG algebra},\ \mathrm{D}^{\mathsf{c}}(A) \simeq \mathcal{T}$$

For fixed $\mathcal{T} \ni G$, when can we guarantee that A is unique up to quasi-iso?

$$A, B\text{: DG algebras such that } H^\bullet(A) = \bigoplus_{i\in\mathbb{Z}} \mathcal{T}(G, \Sigma^i G) = H^\bullet(B)$$

$$(?) \qquad \Longrightarrow \qquad A \overset{\text{q-iso}}{\simeq} B \qquad \Longrightarrow \qquad \mathrm{D}^{\mathsf{c}}(A)_{\mathsf{dg}} \overset{\text{q-eq}}{\simeq} \mathrm{D}^{\mathsf{c}}(B)_{\mathsf{dg}}$$

We work over a perfect field $\mathbf{k}$

e.g. $\operatorname{char} \mathbf{k} = 0$ or $\mathbf{k} = \overline{\mathbf{k}}$

DG enhancements

Pre-triangulated DG categories

DG category = category enriched in cochain complexes of $\mathbf{k}$-vector spaces

Definition (Bondal–Kapranov 1990)

$\mathcal{A}$: ess. small DG category is **(Karoubian) pre-triangulated** if

$$\mathbf{y}\colon \mathcal{A} \hookrightarrow \mathrm{D}^{\mathsf{c}}(\mathcal{A})_{\mathsf{dg}}, \qquad a \longmapsto \mathcal{A}(-,a),$$

induces an equivalence

$$H^0(\mathbf{y})\colon H^0(\mathcal{A}) \xhookrightarrow{\sim} \mathrm{D}^{\mathsf{c}}(\mathcal{A})$$

$\mathcal{A}$: pre-triangulated DG cat. $\implies$ $H^0(\mathcal{A})$ is (canonically) a triangulated cat.

Enhancements of triangulated categories

(Bondal–Kapranov 1990) **DG enhancement** $\mathcal{A}$ **of** $(\mathcal{T}, \Sigma, \triangle)$

- $\mathcal{A}$: pre-triangulated DG category
- $\exists \Phi \colon H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{T}$: equivalence of triangulated categories

$\mathcal{A} \sim \mathcal{B}$ generated by

$$\exists F \colon \mathcal{A} \xrightarrow{\text{quasi-equiv.}} \mathcal{B}$$

$$H^0(F) \colon H^0(\mathcal{A}) \xrightarrow{\sim} H^0(\mathcal{B})$$

$\mathrm{DGE}_3(\mathcal{T}, \Sigma, \triangle)$

Equivalence classes of DG enhancements

$(\mathcal{T}, \Sigma, \triangle)$ admits a **unique DG enhancement** if $\mathrm{DGE}_3(\mathcal{T}, \Sigma, \triangle) = \{*\}$

(Non-)uniqueness of DG enhancements

The following ($\mathbf{k}$-linear) triangulated categories admit a **unique DG enhancement**:

Keller 1994, Lunts–Orlov 2010, Canonaco–Stellari 2018, Canonaco–Neeman–Stellari 2022
All derived and homotopy categories of abelian categories,
certain 'algebro-geometric' derived categories

Muro 2022 ($d=1$), J–Muro 2022 ($d \geq 1$)
Hom-finite + Krull–Schmidt + algebraic tri. cats. with a $d\mathbb{Z}$-cluster tilting object

---

Rizzardo-Van den Bergh 2019, 2020 $\mathbf{k}$-linear triangulated categories with **non-unique DG enhancements** and **without any DG enhancements**

Schlichting 2002, Dugger–Shipley 2007 $\mathbb{Z}$-linear algebraic tri. cats. with **non-unique DG enhancements**

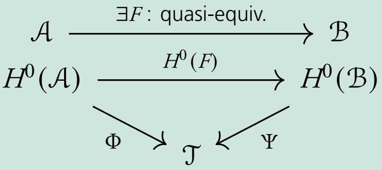
Muro–Schwede–Strickland 2007 $\mathbb{Z}$-linear triangulated categories **without any enhancements** at all (algebraic nor topological)

Strong enhancements of triangulated categories

(Lunts–Orlov 2010) **Strong DG enhancement** $(\mathcal{A}, \Phi)$ **of** $(\mathcal{T}, \Sigma, \triangle)$

- $\mathcal{A}$: pre-triangulated DG category
- $\Phi \colon H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{T}$: equivalence of triangulated categories

$(\mathcal{A}, \Phi) \sim (\mathcal{B}, \Psi)$ generated by



$\mathrm{SDGE}_3(\mathcal{T}, \Sigma, \triangle)$

Equivalence classes of strong DG enhancements

$(\mathcal{T}, \Sigma, \triangle)$ admits a **unique <u>strong</u> DG enhancement** if $\mathrm{SDGE}_3(\mathcal{T}, \Sigma, \triangle) = \{*\}$

Uniqueness of strong enhancements

The following ($\mathbf{k}$-linear) triangulated cats. admit a **unique strong DG enhancement**:

Lunts–Orlov 2010, Canonaco–Stellari 2018, Olander 2020+2022, Li–Pertusi–Zhao 2022
Various 'algebro-geometric' triangulated categories

Chen–Ye 2018, Lorenzin 2022
Bounded derived categories of hereditary abelian categories

Question: Does there exist an algebraic triangulated category with a unique DG enhancement but non-unique strong DG enhancements?

Yes!

DG enhancements in HHA

dZ-cluster tilting subcategories (d≥1)

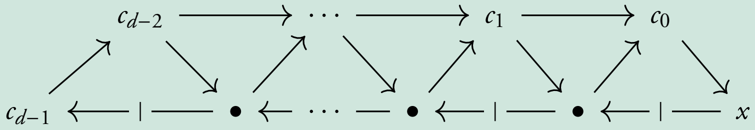
Definition (Iyama–Yoshino 2008, Beligiannis 2015)

$\mathcal{T}$: Hom-finite + Krull–Schmidt triangulated category

$\mathcal{C} = \mathrm{add}(\mathcal{C}) \subseteq \mathcal{T}$ is d-**cluster-tilting** if

- $\forall 0 < i < d, \quad \mathcal{T}(\mathcal{C}, \Sigma^i \mathcal{C}) = 0$
- $\mathcal{T} = \mathcal{C} * \Sigma\mathcal{C} * \cdots * \Sigma^{d-1}\mathcal{C}$

$\forall x \in \mathcal{T} \qquad \exists$



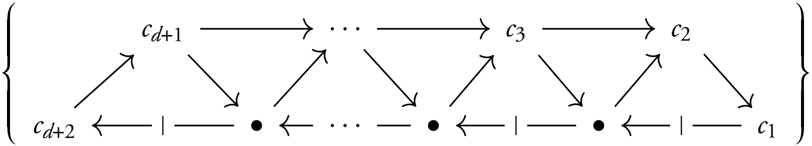
with $c_0, c_1, \ldots, c_{d-1} \in \mathcal{C}$ (<u>without</u> any shifts).

Standard (d+2)-angulated categories

$\mathcal{T}$: Hom-finite + Krull–Schmidt triangulated category

$\mathcal{C} \subseteq \mathcal{T}$: $d\mathbb{Z}$-cluster tilting subcategory = d-cluster tilting + $\mathcal{C} = \Sigma^d(\mathcal{C})$

$\pentagon$ =



Theorem (Geiß–Keller–Oppermann 2013)

The triple $(\mathcal{C}, \Sigma^d, \pentagon)$ is a $(d+2)$-angulated category

$\mathcal{C} \subseteq \mathcal{T}$: $1\mathbb{Z}$-cluster tilting $\Longleftrightarrow$ $\mathcal{C} = \mathcal{T}$

Twisted (d+2)-periodic algebras

$(\mathcal{F}, \Sigma, \pentagon)$: $(d+2)$-angulated category + Hom-finite + Krull–Schmidt

Suppose $\exists c \in \mathcal{F}$ basic object s.t. $\mathrm{add}(c) = \mathcal{F}$ $\rightsquigarrow$ $\Lambda := \mathcal{F}(c, c)$

Theorem (Freyd 1966 + Heller 1968 $d=1$, Geiss–Keller–Opermann 2013 + Green–Snashall–Solberg 2003 + Hanihara 2022)

- Λ : basic Frobenius algebra
- $\exists\ \sigma : \Lambda \xrightarrow{\sim} \Lambda$ algebra automorphism s.t.

$$\Omega_{\Lambda}^{d+2} \cong (-)_{\sigma} : \underline{\mathrm{mod}}(\Lambda) \xrightarrow{\sim} \underline{\mathrm{mod}}(\Lambda)$$

- $\Omega_{\Lambda^e}^{d+2}(\Lambda) \simeq \Lambda_{\sigma}$ in $\underline{\mathrm{mod}}(\Lambda^e)$

We say that Λ is **twisted $(d+2)$-periodic** w.r.t σ

Amiot–Lin (d+2)-angulations

Λ : basic Frobenius algebra & $\sigma\colon \Lambda \xrightarrow{\sim} \Lambda$ & $\mathcal{P}(\Lambda) = \operatorname{proj}(\Lambda)$

$\Sigma := - \otimes_\Lambda \Lambda_{\sigma^{-1}} : \mathcal{P}(\Lambda) \xrightarrow{\sim} \mathcal{P}(\Lambda)$

$\delta \in \operatorname{Ext}^{d+2}_{\Lambda^e}(\Lambda, \Lambda_\sigma) : 0 \to \Lambda_\sigma \to P_{d+1} \to \cdots \to P_1 \to P_0 \to \Lambda \to 0, \quad P_i \in \mathcal{P}(\Lambda^e)$

$\pentagon_\delta = \{Q_{d+2} \to Q_{d+1} \to \cdots \to Q_1 \to \Sigma Q_{d+2}\}$

'Exact sequences in $\mathcal{P}(\Lambda)$ satisfying certain exactness conditions rel δ'

Theorem (Amiot 2008 ($d = 1$), Lin 2019 ($d \geq 1$))

The triple $(\mathcal{P}(\Lambda), \Sigma, \pentagon_\delta)$ is a $(d+2)$-angulated category

J–Muro 2022: Up to equivalence, $\pentagon_\delta$ is independent of the choice of δ

Pre-(d+2)-angulated DG categories

$\mathcal{A}$: DG category is **(Karoubian) pre-(d+2)-angulated** if

$$\mathbf{y}\colon \mathcal{A} \hookrightarrow \mathrm{D}^{\mathsf{c}}(\mathcal{A})_{\mathsf{dg}}, \qquad a \longmapsto \mathcal{A}(-,a),$$

induces an equivalence

$$H^0(\mathbf{y})\colon H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{C} \subseteq \mathrm{D}^{\mathsf{c}}(\mathcal{A})$$

with a $d\mathbb{Z}$-cluster tilting subcategory $\mathcal{C} \subseteq \mathrm{D}^{\mathsf{c}}(\mathcal{A})$

$\mathcal{A}$ pre $(d+2)$-angulated cat. $\implies$ $H^0(\mathcal{A})$ is a std. $(d+2)$-angulated cat.

(Karoubian) pre-$(1+2)$-angulated = (Karoubian) pre-triangulated

Enhancements of (d+2)-angulated categories

DG enhancement $\mathcal{A}$ of $(\mathcal{F}, \Sigma, \pentagon)$

- $\mathcal{A}$: pre-$(d+2)$-angulated DG category
- $\exists \Phi \colon H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{F}$: equivalence of $(d+2)$-angulated categories

$\mathcal{A} \sim \mathcal{B}$ generated by

$$\mathcal{A} \xrightarrow{\exists f\colon \text{quasi-eq}} \mathcal{B}$$

$$H^0(\mathcal{A}) \xrightarrow{H^0(f)} H^0(\mathcal{B})$$

$\mathrm{DGE}_{d+2}(\mathcal{F}, \Sigma, \pentagon)$

Equivalence classes of
DG enhancements

$(\mathcal{F}, \Sigma, \pentagon)$ admits a **unique DG enhancement** if $\mathrm{DGE}_{d+2}(\mathcal{F}, \Sigma, \pentagon) = \{*\}$

Uniqueness of pre-(d+2)-angulated DG enhancements

Λ : basic Frobenius algebra & $\sigma\colon \Lambda \xrightarrow{\sim} \Lambda$ s.t. $\Omega^{d+2}_{\Lambda^e}(\Lambda) \simeq \Lambda_\sigma$

$\Sigma := - \otimes_\Lambda \Lambda_{\sigma^{-1}} : \mathcal{P}(\Lambda) \xrightarrow{\sim} \mathcal{P}(\Lambda)$ & $\pentagon$: Amiot–Lin $(d+2)$-angulation of $(\mathcal{P}(\Lambda), \Sigma)$

Theorem (Muro 2022 ($d = 1$), J–Muro 2022 ($d \geq 1$))

- $(\mathcal{P}(\Lambda), \Sigma, \pentagon)$ admits a DG enhancement and it is moreover unique.
- Up to equivalence,

$$\exists!\ \mathcal{T}\colon \text{ **algebraic** triangulated. cat.,} \qquad (\mathcal{P}(\Lambda), \Sigma, \pentagon) \xrightarrow{\simeq} \mathcal{C} \subseteq \mathcal{T},$$

where $\mathcal{C} \subseteq \mathcal{T}$ is a $d\mathbb{Z}$-cluster tilting subcategory.

Moreover, $\mathcal{T}$ admits a **unique DG enhancement**.

Strong DG enhancements in HHA

Strong enhancements of (d+2)-angulated categories

Strong DG enhancement $(\mathcal{A}, \Phi)$ **of** $(\mathcal{F}, \Sigma, \pentagon)$

- $\mathcal{A}$: pre-$(d+2)$-angulated DG cat
- $\Phi\colon H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{F}$: equivalence of $(d+2)$-angulated categories

$(\mathcal{A}, \Phi) \sim (\mathcal{B}, \Psi)$ generated by

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\exists f\colon \text{quasi-eq}} & \mathcal{B} \\ H^0(\mathcal{A}) & \xrightarrow{H^0(f)} & H^0(\mathcal{B}) \\ & {\scriptstyle \Phi}\searrow \quad \swarrow{\scriptstyle \Psi} & \\ & \mathcal{F} & \end{array}$$

$\mathrm{SDGE}_{d+2}(\mathcal{F}, \Sigma, \pentagon)$

Equivalence classes of strong DG enhancements

$(\mathcal{F}, \Sigma, \pentagon)$ admits a **unique strong DG enhancement** if $\mathrm{SDGE}_{d+2}(\mathcal{F}, \Sigma, \pentagon) = \{*\}$

Pre-triangulated vs pre-(d+2)-angulated enhancements

$\mathcal{T}$: algebraic triangulated category + Hom-finte + Krull–Schmidt

$c \in \mathcal{T}$: $d\mathbb{Z}$-cluster tilting object $\quad\rightsquigarrow\quad \mathcal{C} := \mathrm{add}(c) \subseteq \mathcal{T}$

There is a canonical **restriction map**

$$\mathrm{SDGE}_3(\mathcal{T}, \Sigma, \triangle) \longrightarrow \mathrm{SDGE}_{d+2}(\mathcal{C}, \Sigma^d, \pentagon)$$

$$[\mathcal{A}] \longmapsto [\mathcal{A}_{\mathcal{C}}]$$

$$\begin{array}{ccc} H^0(\mathcal{A}) & \hookleftarrow & H^0(\mathcal{A}_{\mathcal{C}}) \\ \downarrow\wr & & \downarrow\wr \\ \mathcal{T} & \hookleftarrow & \mathcal{C} \end{array}$$

Open question: Is this map is injective or surjective if $d \geq 2$?

Results on strong DG enhancements

The stable centre and the map $\zeta^\times$

Λ : basic Frobenius algebra $\rightsquigarrow$ $\underline{\mathrm{mod}}(\Lambda) = \mathrm{mod}(\Lambda)/[\mathcal{P}(\Lambda)]$

$$\begin{array}{ccccccc}
[\Lambda^e](\Lambda,\Lambda) & \overset{\sim}{\longleftrightarrow} & \mathrm{U}(\Lambda) & & & & \\
\downarrow & & \downarrow & & & & \\
\mathrm{Hom}_{\Lambda^e}(\Lambda,\Lambda) & \overset{\sim}{\longleftrightarrow} & \mathrm{Z}(\Lambda) & \overset{\sim}{\longrightarrow} & \mathrm{Z}(\mathrm{mod}(\Lambda)) & = & \mathrm{End}\left(\mathbf{1}_{\mathrm{mod}(\Lambda)}\right) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\underline{\mathrm{Hom}}_{\Lambda^e}(\Lambda,\Lambda) & \overset{\sim}{\longleftrightarrow} & \underline{\mathrm{Z}}(\Lambda) & \overset{\zeta}{\longrightarrow} & \mathrm{Z}(\underline{\mathrm{mod}}(\Lambda)) & = & \mathrm{End}\left(\mathbf{1}_{\underline{\mathrm{mod}}(\Lambda)}\right) \\
\| & & \uparrow & & \uparrow & & \| \\
\underline{\mathrm{HH}}^0(\Lambda) & & \underline{\mathrm{Z}}(\Lambda)^\times & \overset{\zeta^\times}{\longrightarrow} & \mathrm{Z}(\underline{\mathrm{mod}}(\Lambda))^\times & & \mathrm{HH}^0(\underline{\mathrm{mod}}(\Lambda)) \\
 & & \| & & \| & & \\
 & & \underline{\mathrm{HH}}^0(\Lambda)^\times & & \mathrm{HH}^0(\underline{\mathrm{mod}}(\Lambda))^\times & &
\end{array}$$

Main theorem

$$\Lambda : \text{ basic Frobenius algebra} \quad \& \quad \sigma\colon \Lambda \xrightarrow{\sim} \Lambda \quad \text{s.t.} \quad \Omega^{d+2}_{\Lambda^e}(\Lambda) \simeq \Lambda_\sigma$$

$$\Sigma := - \otimes_\Lambda \Lambda_{\sigma^{-1}} : \mathcal{P}(\Lambda) \xrightarrow{\sim} \mathcal{P}(\Lambda) \quad \& \quad \pentagon : \text{ Amiot–Lin } (d+2)\text{-angulation of } (\mathcal{P}(\Lambda), \Sigma)$$

Theorem (J–Muro 2022)

$\Lambda(\sigma, d) = \bigoplus_{di \in d\mathbb{Z}} \sigma^i \Lambda_1$. There are **bijections:**

$$\begin{array}{ccccc}
\mathrm{SDGE}_{d+2}(\mathcal{P}(\Lambda), \Sigma, \pentagon) & \hookrightarrow & \mathrm{SDGE}_{d+2}(\mathcal{P}(\Lambda), \Sigma) & \xleftrightarrow{\sim} & \mathrm{Aut}(\Lambda(\sigma, d))/{\sim} \\
\wr\updownarrow & & \wr\updownarrow & & \\
\ker \zeta^\times & \hookrightarrow & \underline{Z}(\Lambda)^\times & \xrightarrow{\zeta^\times} & Z(\underline{\mathrm{mod}}(\Lambda))^\times
\end{array}$$

$(\mathcal{P}(\Lambda), \Sigma, \pentagon)$ admits a unique strong DG enhancement $\iff$ $\ker \zeta^\times = 1$

The case d=1 – A complete answer

Λ : basic Frobenius algebra & $\sigma\colon \Lambda \xrightarrow{\sim} \Lambda$ s.t. $\Omega^3_{\Lambda^e}(\Lambda) \simeq \Lambda_\sigma$

$\Sigma := - \otimes_\Lambda \Lambda_{\sigma^{-1}} \colon \mathcal{P}(\Lambda) \xrightarrow{\sim} \mathcal{P}(\Lambda)$ & $\pentagon$: Amiot triangulation of $(\mathcal{P}(\Lambda), \Sigma)$

Theorem (J–Muro 2022)

$\Lambda(\sigma) = \bigoplus_{i\in\mathbb{Z}} {}_{\sigma^i}\Lambda_1$. There are **bijections**:

$$\begin{array}{ccccc}
\mathrm{SDGE}_3(\mathcal{P}(\Lambda), \Sigma, \triangle) & \hookrightarrow & \mathrm{SDGE}_3(\mathcal{P}(\Lambda), \Sigma) & \stackrel{\sim}{\longleftrightarrow} & \mathrm{Aut}(\Lambda(\sigma))/\sim \\
\updownarrow\wr & & \updownarrow\wr & & \\
\ker \zeta^\times & \hookrightarrow & \underline{Z}(\Lambda)^\times & \xrightarrow{\zeta^\times} & Z(\underline{\mathrm{mod}}(\Lambda))^\times
\end{array}$$

$(\mathcal{P}(\Lambda), \Sigma, \pentagon)$ admits a unique <u>strong</u> DG enhancement $\Leftrightarrow$ $\ker \zeta^\times = 1$

The algebra of dual numbers – An explicit example

$\Lambda = \mathbf{k}[\varepsilon]$: algebra of dual numbers & $\sigma\colon \varepsilon \longmapsto -\varepsilon$

$\Sigma = - \otimes_\Lambda {}_1\Lambda_{\sigma^{-1}} \colon \mathcal{P}(\Lambda) \xrightarrow{\sim} \mathcal{P}(\Lambda)$ & $\triangle$: Amiot triangulation of $(\mathcal{P}(\Lambda), \Sigma)$

$$\begin{array}{ccc}
Z(\Lambda) & \overset{\sim}{\longleftrightarrow} & Z(\operatorname{mod}(\Lambda)) \\
\twoheaddownarrow & & \downarrow \\
\underline{Z}(\Lambda) & \longrightarrow & Z(\underline{\operatorname{mod}}(\Lambda)) \\
\uparrow & & \uparrow \\
\underline{Z}(\Lambda)^\times & \xrightarrow{\zeta^\times} & Z(\underline{\operatorname{mod}}(\Lambda))^\times
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Lambda & \overset{\sim}{\longleftrightarrow} & \Lambda \\
\twoheaddownarrow & & \downarrow \\
\Lambda/(2\varepsilon) & \xrightarrow{\zeta} & \mathbf{k} \\
\uparrow & & \uparrow \\
\underline{Z}(\Lambda)^\times & \xrightarrow{\zeta^\times} & \mathbf{k}^\times
\end{array}$$

$\operatorname{char}(\mathbf{k}) \neq 2 \implies \zeta = \mathrm{id} \implies \mathrm{SDGE}_3(\mathcal{P}(\Lambda), \Sigma, \triangle) \leftrightarrow \ker \zeta^\times = 1$

$\operatorname{char}(\mathbf{k}) = 2 \implies \zeta = \mathrm{aug} \implies \mathrm{SDGE}_3(\mathcal{P}(\Lambda), \Sigma, \triangle) \leftrightarrow \ker \zeta^\times = 1 + (\varepsilon)$

LUND
UNIVERSITY

References

[BK90] A. I. Bondal and M. M. Kapranov. "Framed triangulated categories". *Mat. Sb.* 181.5 (1990), pp. 669–683.

[CNS22] Alberto Canonaco, Amnon Neeman, and Paolo Stellari. "Uniqueness of enhancements for derived and geometric categories". *Forum Math. Sigma* 10 (2022), Paper No. e92, 65. DOI: `10.1017/fms.2022.82`.

[CS18] Alberto Canonaco and Paolo Stellari. "Uniqueness of dg enhancements for the derived category of a Grothendieck category". *J. Eur. Math. Soc. (JEMS)* 20.11 (2018), pp. 2607–2641. DOI: `10.4171/JEMS/820`.

[CY18] Xiao-Wu Chen and Yu Ye. "The $\mathbf{D}$-standard and $\mathbf{K}$-standard categories". *Adv. Math.* 333 (2018), pp. 159–193. DOI: `10.1016/j.aim.2018.05.032`.

[DS07] Daniel Dugger and Brooke Shipley. "Topological equivalences for differential graded algebras". *Adv. Math.* 212.1 (2007), pp. 37–61. DOI: `10.1016/j.aim.2006.09.013`.

[JM22] Gustavo Jasso and Fernando Muro. *The Derived Auslander–Iyama Correspondence*. With an appendix by Bernhard Keller. 2022. arXiv: `2208.14413`.

[Kel94] Bernhard Keller. "Deriving DG categories". *Ann. Sci. École Norm. Sup. (4)* 27.1 (1994), pp. 63–102.

[LO10] Valery A. Lunts and Dmitri O. Orlov. "Uniqueness of enhancement for triangulated categories". *J. Amer. Math. Soc.* 23.3 (2010), pp. 853–908. DOI: `10.1090/S0894-0347-10-00664-8`.

[Lor22] Antonio Lorenzin. *Formality and strongly unique enhancements*. 2022. arXiv: `2204.09527`.

[LPZ22] Chunyi Li, Laura Pertusi, and Xiaolei Zhao. *Derived categories of hearts on Kuznetsov components*. 2022. arXiv: `2203.13864`.

[Ola20] Noah Olander. *Orlov's Theorem in the Smooth Proper Case*. 2020. arXiv: `2006.15173`.

[Ola22] Noah Olander. *Resolutions, Bounds, and Dimensions for Derived Categories of Varieties*. Thesis (Ph.D.)–Columbia University. ProQuest LLC, Ann Arbor, MI, 2022, p. 49. ISBN: 979-8426-81551-3.

[RV19] Alice Rizzardo and Michel Van den Bergh. "A note on non-unique enhancements". *Proc. Amer. Math. Soc.* 147.2 (2019), pp. 451–453. DOI: `10.1090/proc/14065`.

[RV20] Alice Rizzardo and Michel Van den Bergh. "A k-linear triangulated category without a model". *Ann. of Math. (2)* 191.2 (2020), pp. 393–437. DOI: `10.4007/annals.2020.191.2.3`.

[Sch02] Marco Schlichting. "A note on K-theory and triangulated categories". *Invent. Math.* 150.1 (2002), pp. 111–116. DOI: `10.1007/s00222-002-0231-1`.