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Quivers, quivers and more quivers

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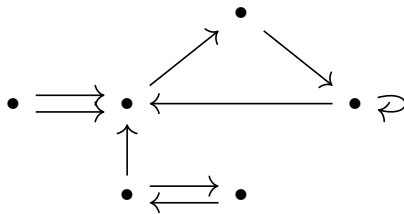


What is a quiver?

quiv·er (noun) a case for carrying or holding arrows



quiv·er (math.) a directed graph



But ... why quivers?

Theorem: V, W : vector spaces

$$V \cong W \iff \dim V = \dim W$$

Recall: Two $m \times n$ matrices A and B are **equivalent** if

$$B = TAS^{-1}, \quad T \in GL(m), S \in GL(n)$$

and we write $A \sim B$

Theorem:

$$A \sim B \iff \text{rank } A = \text{rank } B$$

A commutative diagram with three nodes: V at the top left, W at the top right, and \mathbb{C}^n at the bottom center. An arrow labeled α points from V to \mathbb{C}^n . An arrow labeled β points from W to \mathbb{C}^n . A top arrow labeled $\beta^{-1} \circ \alpha$ points from V to W .

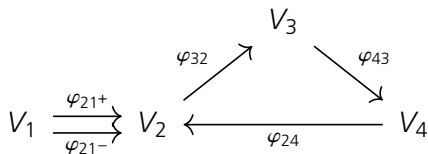
α, β : choice of bases

A commutative diagram with four nodes: \mathbb{C}^n at the top left, \mathbb{C}^m at the top right, \mathbb{C}^n at the bottom left, and \mathbb{C}^m at the bottom right. A top arrow labeled A points from the top-left \mathbb{C}^n to the top-right \mathbb{C}^m . A bottom arrow labeled B points from the bottom-left \mathbb{C}^n to the bottom-right \mathbb{C}^m . A left vertical arrow labeled S points from the top-left \mathbb{C}^n to the bottom-left \mathbb{C}^n . A right vertical arrow labeled T points from the top-right \mathbb{C}^m to the bottom-right \mathbb{C}^m .

Quiver representations

A **representation** \underline{V} of a quiver Q consists of

1. a vector space V_x for every vertex x of Q
2. a linear map $\varphi_a: V_x \rightarrow V_y$ for every $a: x \rightarrow y$ in Q



A representation of $Q = \bullet$ is a vector space V

A representation of $Q = 1 \rightarrow 2$ is a linear map $\varphi: V \rightarrow W$

A representation of

$$Q = \bullet \rightrightarrows \bullet$$

is a linear map $\varphi: V \rightarrow V$

Classification problems in linear algebra

Two representations $\underline{V}, \underline{W}$ of a quiver Q are **isomorphic** if

$$\exists f_x: V_x \xrightarrow{\sim} W_x, \quad x: \text{vertex of } Q$$

s.t. for every $a: x \rightarrow y$ in Q

$$\begin{array}{ccc} V_x & \xrightarrow{\varphi_a} & V_y \\ \downarrow f_x & & \downarrow f_y \\ W_x & \xrightarrow{\psi_a} & W_y \end{array}$$

$$f_y \circ \varphi_a = \psi_a \circ f_x$$

$$Q = 1 \rightarrow 2 \quad B = TAS^{-1}$$

$$\begin{array}{ccccc} \underline{U}: & \mathbb{C}^n & \xrightarrow{A} & \mathbb{C}^m \\ \downarrow & \downarrow S & & \downarrow T \\ \underline{V}: & \mathbb{C}^n & \xrightarrow{B} & \mathbb{C}^m \end{array}$$

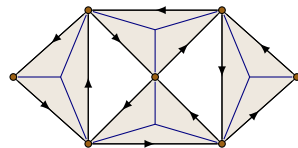
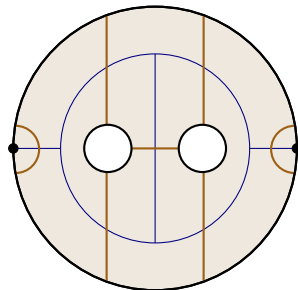
$$Q = \bullet \curvearrowright \quad B = SAS^{-1} \quad \text{JNF}$$

$$\begin{array}{ccccc} \underline{U}: & \mathbb{C}^n & \xrightarrow{A} & \mathbb{C}^n \\ \downarrow & \downarrow S & & \downarrow S \\ \underline{V}: & \mathbb{C}^n & \xrightarrow{B} & \mathbb{C}^n \end{array}$$

Quivers are everywhere!

- Algebraic Geometry
- Algebraic Topology
- Combinatorics
- Commutative Algebra
- Lie Theory
- Representation Theory
- Symplectic Geometry

... even in Topological Data Analysis!





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Extra: Normal forms and quiver representations

$$A: \mathbb{C}^n \longrightarrow \mathbb{C}^m \quad \longleftrightarrow \quad [1]^{\oplus \text{rank}(A)} \oplus \mathbb{=}^{\oplus \text{nullity}(A)} \oplus []^{\oplus \text{corank}(A)}$$

$$\begin{array}{c}
 \overbrace{\hspace{1.5cm}}^{\text{rank}(A)} \quad \overbrace{\hspace{1.5cm}}^{\text{nullity}(A)} \\
 \left(\begin{array}{ccc|ccc}
 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 & \ddots & \ddots & & & & \\
 0 & & \ddots & & & & \\
 & \ddots & & & & & \\
 & & \ddots & & & & \\
 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
 \hline
 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & \ddots & & & & & \\
 0 & \cdots & 0 & 0 & 0 & \cdots & 0
 \end{array} \right) \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{rank}(A) \\ \\ \\ \\ \\ \\ \\ \text{corank}(A) \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 [] \\
 \downarrow_i \\
 [1] \\
 \downarrow_p \\
 \mathbb{=}
 \end{array}
 \quad
 \begin{array}{ccc}
 (0 \longrightarrow \mathbb{C}) & & \\
 \downarrow & & \downarrow_1 \\
 (\mathbb{C} \xrightarrow{1} \mathbb{C}) & & \\
 \downarrow_1 & & \downarrow \\
 (\mathbb{C} \longrightarrow 0) & &
 \end{array}$$