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# Quivers, quivers and more quivers

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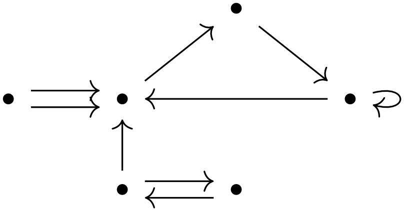
# What is a quiver?

**quiv·er** (noun) a case for carrying or holding arrows

**quiv·er** (math.) a directed graph



# But … why quivers?

**Theorem:** $V, W$: vector spaces

$$V \cong W \iff \dim V = \dim W$$

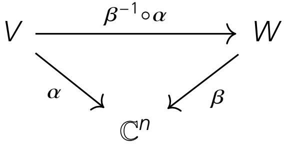
Recall: Two $m \times n$ matrices $A$ and $B$ are **equivalent** if

$$B = TAS^{-1}, \quad T \in \mathrm{GL}(m),\ S \in \mathrm{GL}(n)$$

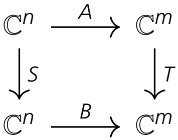
and we write $A \sim B$

**Theorem:**

$$A \sim B \iff \operatorname{rank} A = \operatorname{rank} B$$



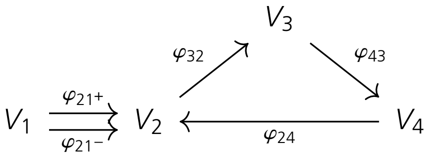

$\alpha, \beta$: choice of bases



# Quiver representations

A **representation** $\underline{V}$ of a quiver $Q$ consists of

1. a vector space $V_x$ for every vertex $x$ of $Q$
2. a linear map $\varphi_a \colon V_x \to V_y$ for every $a \colon x \to y$ in $Q$




A representation of $Q = \bullet$ is a vector space $V$

A representation of $Q = 1 \longrightarrow 2$ is a linear map $\varphi \colon V \to W$

A representation of

$$Q = \bullet \circlearrowleft$$

is a linear map $\varphi \colon V \to V$

# Classification problems in linear algebra

Two representations $\underline{V}, \underline{W}$ of a quiver $Q$ are **isomorphic** if

$$\exists\ f_x \colon V_x \xrightarrow{\sim} W_x, \quad x : \text{vertex of } Q$$

s.t. for every $a \colon x \to y$ in $Q$

$$\begin{array}{ccc} V_x & \xrightarrow{\varphi_a} & V_y \\ \downarrow{\scriptstyle f_x} & & \downarrow{\scriptstyle f_y} \\ W_x & \xrightarrow{\psi_a} & W_y \end{array}$$

$$f_y \circ \varphi_a = \psi_a \circ f_x$$

$$Q = 1 \to 2 \qquad B = TAS^{-1}$$

$$\begin{array}{ccccc} \underline{U}\colon & & \mathbb{C}^n & \xrightarrow{A} & \mathbb{C}^m \\ \downarrow & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle T} \\ \underline{V}\colon & & \mathbb{C}^n & \xrightarrow{B} & \mathbb{C}^m \end{array}$$

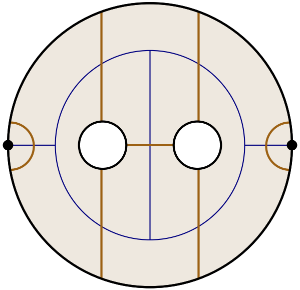
$$Q = \bullet \circlearrowleft \qquad B = SAS^{-1} \quad \text{JNF}$$

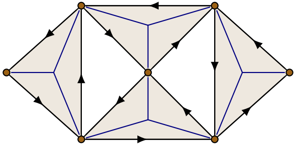
$$\begin{array}{ccccc} \underline{U}\colon & & \mathbb{C}^n & \xrightarrow{A} & \mathbb{C}^n \\ \downarrow & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle S} \\ \underline{V}\colon & & \mathbb{C}^n & \xrightarrow{B} & \mathbb{C}^n \end{array}$$

# Quivers are everywhere!

- Algebraic Geometry
- Algebraic Topology
- Combinatorics
- Commutative Algebra
- Lie Theory
- Representation Theory
- Symplectic Geometry

… even in Topological Data Analysis!





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# Extra: Normal forms and quiver representations

$$A\colon \mathbb{C}^n \longrightarrow \mathbb{C}^m \qquad \longleftrightarrow \qquad [1]^{\oplus\,\mathrm{rank}(A)} \oplus \square^{\oplus\,\mathrm{nullity}(A)} \oplus [\,]^{\oplus\,\mathrm{corank}(A)}$$

$$\begin{array}{c} \overbrace{\phantom{xxxxxxxx}}^{\mathrm{rank}(A)}\;\overbrace{\phantom{xxxxxx}}^{\mathrm{nullity}(A)} \\ \left(\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\0\\0\\0\end{array}}\right\}\mathrm{rank}(A) \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\}\mathrm{corank}(A) \end{array} \end{array}$$

$$\begin{array}{c} [\,] \\ \downarrow{\scriptstyle i} \\ [1] \\ \downarrow{\scriptstyle p} \\ \square \end{array} \qquad\qquad \begin{array}{ccc} (0 & \longrightarrow & \mathbb{C}) \\ \downarrow & & \downarrow{\scriptstyle 1} \\ (\mathbb{C} & \xrightarrow{1} & \mathbb{C}) \\ \downarrow{\scriptstyle 1} & & \downarrow \\ (\mathbb{C} & \longrightarrow & 0) \end{array}$$