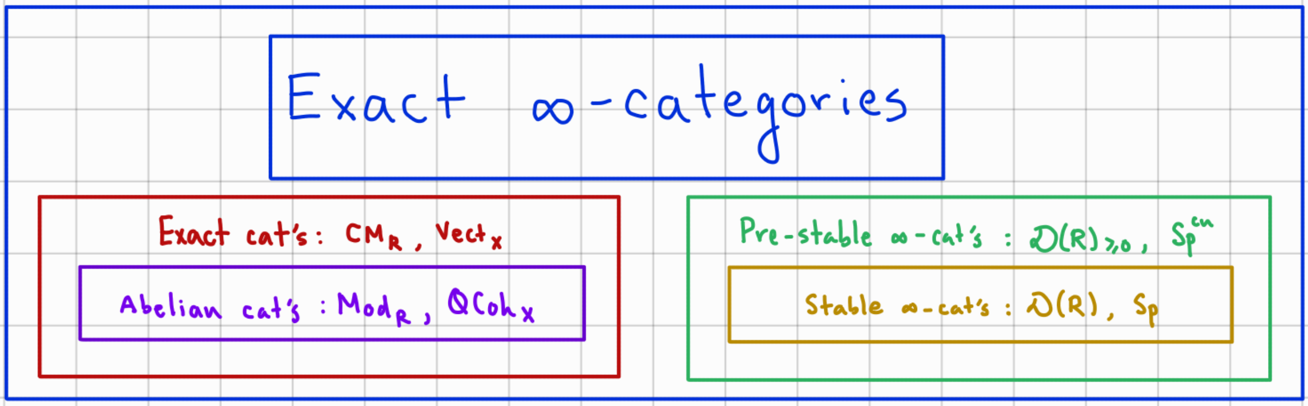
2023-11-20 • 2023-11-24 @ CIRM Luminy / JASSO

Exact ∞-categories



§ Motivation (not historically accurate)

$\mathcal{A}$: abelian cat. , $\mathcal{E} \subseteq$ extension-closed subcategory

$$\mathcal{S} = \{ X \rightarrowtail Y \twoheadrightarrow Z \text{ s.e.s. in } \mathcal{A} \mid X, Y, Z \in \mathcal{E} \}$$

(Quillen 1972) Axiomatisation of the properties of $(\mathcal{E}, \mathcal{S})$

$\rightsquigarrow$ Exact category

Gabriel-Quillen Embedding Thm $(\underset{\text{ess. small}}{\mathcal{E}}, \mathcal{S})$: exact cat.

$\Longrightarrow$ $\exists\, i : \mathcal{E} \hookrightarrow \mathcal{A}$: abelian cat s.t. $i(\mathcal{E}) \subseteq \mathcal{A}$ is extension-closed and i preserves and reflects admissible exact sequences

$(\mathcal{E}, \mathcal{S})$: exact cat $\rightsquigarrow$ $\mathcal{E} \overset{\eta}{\hookrightarrow} \mathcal{D}^b(\mathcal{E}, \mathcal{S})$: tri. cat & $\eta(\mathcal{E})$: extension closed

$\mathcal{T}$: triangulated cat., $\mathcal{E} \subseteq$ extension-closed subcategory

$$\mathcal{S} = \{ X \to Y \to Z \to X[1] \text{ ex. tri in } \mathcal{T} \mid X, Y, Z \in \mathcal{E} \}$$

(Nakaoka - Palu 2019) Axiomatisation of the properties of $(\mathcal{E}, \mathcal{S}, \operatorname{Ext}^1_{\mathcal{T}}|_{\mathcal{E}})$

$\rightsquigarrow$ Extriangulated categories

Q Is there an analogue of the Gabriel-Quillen Embedding Thm for extri. cat's?

We will give a partial answer (Klemenc 2022) leveraging the theory of $\boxed{\infty\text{-categories}}$

§ Crash-course on ∞-category theory

$\mathcal{C}$: 1-cat $\rightsquigarrow$ $\forall X, Y \in \mathcal{C}$, $\mathcal{C}(X,Y)$: set of morphisms $X \to Y$

- Associativity + Unitality $\rightsquigarrow$ $\mathcal{C}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow$ Set functor

- Universal property : $\mathcal{C}(W, X \times_Z Y) \xrightarrow{\sim} \mathcal{C}(W,X) \times_{\mathcal{C}(W,Z)} \mathcal{C}(W,Y)$, $f \longmapsto (p_X \circ f, p_Y \circ f)$

i.e
$$\begin{tikzcd} W \arrow[r, "\varphi"] \arrow[d, "\psi"'] \arrow[dr, phantom, "="] & X \arrow[d, "u"] \\ Y \arrow[r, "v"'] & Z \end{tikzcd}$$
$\rightsquigarrow$ $\{ f : W \to X \times_Z Y \mid (p_X \circ f, p_Y \circ f) = (\varphi, \psi) \} \cong * \longleftarrow$ singleton

Problem $\mathcal{A}$: additive cat. (e.g. $\mathcal{A} = \mathrm{Mod}_R$, R : ring)

$K(\mathcal{A})$: cat. of cochain complexes in $\mathcal{A}$ up to homotopy

not unique in general

$K(\mathcal{A})$:
$$\begin{tikzcd} & W \arrow[dl, dashed, "\bar{a}"'] \arrow[d, "a"] \arrow[dr, dashed, "0"] & \\ \mathrm{cocone}(f) \arrow[r, "g"'] & X \arrow[r, "f"'] & Y \end{tikzcd}$$
$g\bar{a} = a$,
$$\begin{tikzcd} (W, Y[-1]) \arrow[r] & (W, \mathrm{cocone}(f)) \arrow[r, "g \circ ?"] & (W, X) \arrow[r, "f \circ ?"] & (W, Y) \\ & \bar{a} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & a \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & 0 \end{tikzcd}$$

Solution Replace sets / abelian groups by $\boxed{\text{"richer"}}$ mathematical objects

(Bondal-Kapranov 1990) Set $\rightsquigarrow$ Ch(Ab) : cochain complexes of abelian groups

abstract inverse shift, shift of cochain complexes

e.g. $\mathcal{H}om(x, \Omega y) \xrightarrow[\text{q iso}]{\sim} \mathcal{H}om(x, y)[-1]$

1-cat's $\rightsquigarrow$ dg / A_∞-categories $<$
- Advantages: Explicit formulas, adapted to "algebra & geometry"
- Disadvantages: Explicit formulas, adapted to "algebra & geometry"

Set $\rightsquigarrow$ Grp_∞ : ∞-groupoids $\simeq$ homotopy types

1-cat's $\rightsquigarrow$ ∞-categories $<$
- Advantages: No formulas, adapted to "topology"
- Disadvantages: No formulas, adapted to "topology"

Why ∞-groupoids ? Finer invariants, e.g. Khovanov spectra (Lipshitz-Sarkar 2014)

DISCLAIMER Unless noted otherwise, unattributed results are due to Lurie (at least in the form we present)

Today Mechanics of ∞-category theory (after Joyal, Lurie, ...)

(1) There is an ∞-category $\textcircled{\mathrm{Gpd}_\infty}$ whose objects are (small) ∞-groupoids

Grothendieck's Homotopy Hypothesis (simplified version)

- X : top. space $\longmapsto \Pi_\infty(X)$: fundamental ∞-groupoid
- Every ∞-groupoid arises in this way

$\left\{ \begin{array}{l} \text{dim 0: points in } X \\ \text{dim 1: paths in } X \\ \text{dim 2: homotopies btwn paths in } X \\ \vdots \end{array} \right.$

$X \in \mathrm{Gpd}_\infty \rightsquigarrow \pi_0(X)$: set of path connected components / iso classes

$\overset{\rotatebox{90}{$\in$}}{x}$, $k \geq 1 \rightsquigarrow \pi_k(X, x)$: k-th homotopy group (abelian for $k \geq 2$)

Whitehead's Thm $f : X \to Y$ in Gpd_∞ TFAE

(a) f is an isomorphism

(b) • $\pi_0(f) : \pi_0(X) \xrightarrow{\sim} \pi_0(Y)$ is a bijection

• $\forall x \in X$, $\forall k \geq 1$, $\pi_1(f) : \pi_1(X, x) \longrightarrow \pi_1(Y, f(x))$ group isomorphism

$$\begin{tikzcd} \mathrm{Set} \simeq \{ X \in \mathrm{Gpd}_\infty \mid \forall x \in X, \forall k > 0,\ \pi_k(X,x) = 0 \} \subseteq \mathrm{Gpd}_\infty \arrow[d, hook] \\ \textcircled{\mathrm{Gpd}} \simeq \{ X \in \mathrm{Gpd}_\infty \mid \forall x \in X, \forall k > 1,\ \pi_k(X,x) = 0 \} \subseteq \mathrm{Gpd}_\infty \end{tikzcd}$$

cat. of (small) groupoids

$$\begin{tikzcd} X, Y \in \mathrm{Gpd}_\infty \arrow[d, squiggly] \arrow[r, squiggly] & \mathrm{Map}(X, Y) : \infty\text{-groupoid of maps } X \to Y \text{ (functorial)} \\ X \times Y \in \mathrm{Gpd}_\infty \arrow[r, squiggly] & \mathrm{Map}(X \times Y, Z) \overset{\sim}{\leftrightarrows} \mathrm{Map}(X, \mathrm{Map}(Y, Z)) : \text{currying adjunction} \end{tikzcd}$$

$* = \Pi_\infty(pt) \in \mathrm{Gpd}_\infty$: final ∞-groupoid

"homotopy singleton"

X : ∞-groupoid is $\boxed{\text{contractible}}$ if $X \xrightarrow{\sim} *$ is an isomorphism. Equivalently:

- $\pi_0(X)$: singleton
- $\exists x \in X$ s.t. $\forall k \geq 1$ $\pi_k(X, x) = 0$

(2) $\mathcal{C}$: ∞-category $\rightsquigarrow$ $\forall X, Y \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(X,Y)$: ∞-groupoid of maps $X \to Y$ (functorial)

$\downarrow$

$\mathrm{Ho}(\mathcal{C})$: homotopy category $\mathrm{Ho}(\mathcal{C})(X,Y) := \pi_0(\mathrm{Map}_{\mathcal{C}}(X,Y)) \in \mathrm{Set}$

$* \in \mathcal{C}$: final object if $\forall X \in \mathcal{C}$ $\mathrm{Map}_{\mathcal{C}}(X,*)$ is contractible ($\Rightarrow * \in \mathrm{Ho}(\mathcal{C})$: final)

Warning $* \in \mathrm{Ho}(\mathcal{C})$: final object $\not\Rightarrow$ $* \in \mathcal{C}$: final object

(3) There is an ∞-category cat_∞ whose objects are the small ∞-cat's.

$$\mathrm{cat} \simeq \{ \mathcal{C} \in \mathrm{cat}_\infty \mid \forall X, Y \in \mathcal{C},\ \mathrm{Map}_{\mathcal{C}}(X,Y) \in \mathrm{Set} \}$$

Joyal's Thm

$$\mathrm{Gpd}_\infty \overset{\downarrow}{=} \{ \mathcal{C} \in \mathrm{cat}_\infty \mid \mathrm{Ho}(\mathcal{C}) : \text{groupoid} \} \subseteq \mathrm{cat}_\infty$$

$\mathcal{C}, \mathcal{D}$: ∞-cat's $\rightsquigarrow$ $\mathrm{Fun}(\mathcal{C},\mathcal{D})$: ∞-cat of functors (ignoring size issues)

$\mathcal{C} \times \mathcal{D}$: ∞-cat $\rightsquigarrow$ $\mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \overset{\sim}{\Leftrightarrow} \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\mathcal{D}, \mathcal{C}))$ currying adjunction

A : small cat. & $\mathcal{C}$: ∞-cat. $\rightsquigarrow$ $\mathrm{Fun}(A,\mathcal{C})$: ∞-cat of coherent diagrams $A \to \mathcal{C}$

e.g.

$$A = \left\{ \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=3, "="] & & 2 \end{tikzcd} \right\} \xrightarrow{X} \mathcal{C} , \quad \begin{tikzcd} & X_1 \arrow[dr, "g"] & \\ X_0 \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "\alpha"] & & X_2 \end{tikzcd} \quad \text{“}\alpha : g \circ f \simeq h\text{”} \Rightarrow gf = h \text{ in } \mathrm{Ho}(\mathcal{C})$$

Warning $\mathrm{Ho}(\mathrm{Fun}(A,\mathcal{C})) \longrightarrow \mathrm{Fun}(A, \mathrm{Ho}(\mathcal{C}))$ is not an equivalence in general

(Lurie, Faonte 2017) $\mathcal{A}$: dg / A_∞-cat $\longmapsto$ $\hat{\mathcal{A}}$: ∞-category (dg / A_∞-nerve)

$$\mathrm{hom}_{\mathcal{A}}(X,Y) \in \mathrm{Ch}(\mathrm{Mod}_{\Bbbk}) \quad / \quad \mathrm{Map}_{\hat{\mathcal{A}}}(X,Y) \in \mathrm{Grp}_\infty$$

$$H^0(\mathrm{hom}_{\mathcal{A}}(X,Y)) \cong \pi_0(\mathrm{Map}_{\hat{\mathcal{A}}}(X,Y))$$

$$H^{<0}(\mathrm{hom}_{\mathcal{A}}(X,Y)) \cong \pi_{>0}(\mathrm{Map}_{\hat{\mathcal{A}}}(X,Y), 0)$$

soft truncation

Warning ∞-groupoids do not have "negative homotopy groups" : $\hat{\mathcal{A}} = \widehat{(\tau^{\leq 0}\mathcal{A})}$

(5) Robust theory including limits, colimits, adjunctions, ...

Universal property: $\mathrm{Map}_{\mathcal{C}}(W, X \times_Z Y) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}}(W,X) \times_{\mathrm{Map}_{\mathcal{C}}(W,Z)} \mathrm{Map}_{\mathcal{C}}(W,Y)$ in Gpd_∞

e.g. $\star$:
$$\begin{tikzcd} W \arrow[r, "\varphi"] \arrow[d, "\psi"'] \arrow[dr] & X \arrow[d, "u"] \\ Y \arrow[r, "v"'] & Z \end{tikzcd}$$

$$\begin{tikzcd} \text{"}W \to X \times_Z Y \text{ compatible with } \star\text{"} \arrow[r] \arrow[d, "2"] \arrow[dr, phantom, "\text{PB in } \mathrm{Gpd}_\infty"] & \mathrm{Map}_{\mathcal{C}}(W, X \times_Z Y) \arrow[d, "2"] \\ * \arrow[r, "\star"'] & \mathrm{Map}_{\mathcal{C}}(W,X) \times_{\mathrm{Map}_{\mathcal{C}}(W,Z)} \mathrm{Map}_{\mathcal{C}}(W,Y) \end{tikzcd}$$

Warning $\mathcal{C} \xrightarrow{\text{can}} \mathrm{Ho}(\mathcal{C})$ preserves (co)products but not arbitrary (co)limits in general since $\pi_0 : \mathrm{Gpd}_\infty \to \mathrm{Set}$ does not preserve arbitrary limits.

(6) $\mathcal{C}$: ∞-cat & W : class of maps in $\mathcal{C}$ $\rightsquigarrow$ $\gamma : \mathcal{C} \to \mathcal{C}[W^{-1}]$: localisation at W

$\forall \mathcal{D}$: ∞-cat $\quad \gamma^* : \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}_W(\mathcal{C}, \mathcal{D})$ is an equivalence

↑ functors that invert maps in W

$\mathrm{Ho}(\mathcal{C})[W^{-1}] \xrightarrow{\sim} \mathrm{Ho}(\mathcal{C}[W^{-1}])$ equivalence of 1-cat's

Warning $\mathcal{C}[W^{-1}] \longrightarrow \mathrm{Ho}(\mathcal{C}[W^{-1}])$ is not an equivalence in general

$\mathcal{G}$: Grothendieck cat. (e.g. $\mathcal{G} = \mathrm{Mod}_R$, R: ring)

∞-cat localisation

$$\begin{tikzcd} \mathrm{Ch}(\mathcal{G}) \arrow[r] \arrow[dr] & \mathrm{Ch}(\mathcal{G})[\mathrm{qiso}^{-1}] =: \mathcal{D}(\mathcal{G}) \arrow[d, "\leftarrow \text{not an equivalence}"] \\ & \mathrm{Ho}(\mathrm{Ch}(\mathcal{G})[\mathrm{qiso}^{-1}]) = D(\mathcal{G}) \end{tikzcd}$$

1-cat localisation

$\mathcal{D}(\mathcal{G})$: derived ∞-cat of $\mathcal{G}$ — stable ∞-cat.

$\downarrow$ Ho

$D(\mathcal{G})$: derived cat of $\mathcal{G}$ — triangulated cat.

- $\pi_0(\mathrm{Map}_{\mathcal{D}(\mathcal{G})}(X,Y)) \cong \mathrm{Hom}_{D(\mathcal{G})}(X,Y)$
- $\pi_{>0}(\mathrm{Map}_{\mathcal{D}(\mathcal{G})}(X,Y)) \cong \pi_0(\mathrm{Map}_{\mathcal{D}(\mathcal{G})}(X, Y[<0])) \cong \overbrace{\mathrm{Hom}_{\mathcal{D}(\mathcal{G})}(X, Y[<0])}^{\text{negative extensions!}}$

$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \quad \overset{\gamma}{\longmapsto}$
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$
: biCartesian square in $\mathcal{D}(\mathcal{G})$

s.e.s. in $\mathrm{Ch}(\mathcal{G})$

← has a universal property!

= PO + PB

DISCLAIMER Unless noted otherwise, unattributed results are due to Lurie (at least in the form we present)

§ Stable ∞-categories

Def $\mathcal{C}$: ∞-cat is stable if

(0) $\exists\, 0 \in \mathcal{C}$: zero object

(1) $\forall f: X \to Y$ in $\mathcal{C}$ $\exists$ $$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd}$$ & $\exists$ $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "PO"] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$ $W := \mathrm{fib}(f)$, $Z := \mathrm{cofib}(f)$

(2) A square $$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$ in $\mathcal{C}$ is PO $\Longleftrightarrow$ it is PB } fibre-cofibre sequences

Thm $\mathcal{C}$: stable $\Longrightarrow$ $(\mathrm{Ho}(\mathcal{C}), \Sigma, \Delta)$: triangulated cat.

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd} \quad \& \quad \begin{tikzcd} \Omega Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$ $\rightsquigarrow$ $X \simeq \Omega\Sigma X$, $Y \simeq \Sigma\Omega X$ $\rightsquigarrow$ $\Sigma : \mathcal{C} \overset{\sim}{\longleftrightarrow} \mathcal{C} : \Omega$

$$\Delta = \left\{ \begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & \mathrm{cofib}(f) \arrow[r] & \Sigma X \end{tikzcd} \right\}$$: exact triangles in $\mathrm{Ho}(\mathcal{C})$

Warning $$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$ bicartesian in $\mathcal{C}$ $\not\Longrightarrow$ $$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$ bicartesian in $\mathrm{Ho}(\mathcal{C})$ (e.g. $$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$ in $\mathcal{C}$, $$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$ in $\mathrm{Ho}(\mathcal{C})$)

(co)limits in $\mathrm{Fun}(A, \mathcal{C})$ are computed pointwise

Prop A : small cat & $\mathcal{C}$: stable ∞-cat $\Longrightarrow$ $\mathrm{Fun}(A, \mathcal{C})$: stable ∞-cat

Warning $\mathrm{Ho}(\mathrm{Fun}(A, \mathcal{C}))$ is triangulated but $\mathrm{Ho}(\mathrm{Fun}(A, \mathcal{C})) \not\simeq \mathrm{Fun}(A, \mathrm{Ho}(\mathcal{C}))$

e.g. k : field, $\mathrm{Ho}(\mathrm{Fun}(1 \to 2, \mathcal{D}(k))) \simeq \underbrace{D(\mathrm{Fun}(1 \to 2, \mathrm{Mod}(k)))}_{\text{not abelian}} \not\simeq \mathrm{Fun}(1 \to 2, \underbrace{D(k)}_{\cong\, \mathrm{Mod}^{\mathbb{Z}}(k)\,:\ \text{abelian}})$

Def $\mathcal{C}, \mathcal{D}$: stable ∞-cat's.

$F : \mathcal{C} \to \mathcal{D}$ is exact if $F(0) \simeq 0$ and preserves fibre-cofibre sequences

Rmk $F : \mathcal{C} \to \mathcal{D}$ exact $\Longrightarrow$ $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ triangle functor

exact functors

Prop $\mathcal{C}, \mathcal{D}$: stable ∞-cat's $\Longrightarrow$ $\mathrm{Fun}^{ex}(\mathcal{C}, \mathcal{D})$: stable ∞-cat

Prop (e.g. Cisinski 2019) $(\mathcal{A}, \mathcal{S})$: Frobenius exact cat. $W_{\mathcal{S}} = \{ f \mid [f] \text{ is iso in } \mathcal{A}/[\mathcal{S}\text{-proj}_{\mathcal{A}}] \}$

$\Longrightarrow$ $\underline{\mathcal{A}}_{\mathcal{S}} := \mathcal{A}[W_{\mathcal{S}}^{-1}]_{\infty}$: stable ∞-cat and

$\gamma : \mathcal{A} \longrightarrow \underline{\mathcal{A}}_{\mathcal{S}}$ sends $(X \rightarrowtail Y \twoheadrightarrow Z)$ $\overset{\gamma}{\longmapsto}$
$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$
fibre-cofibre sequences in $\underline{\mathcal{A}}_{\mathcal{S}}$

admissible s.e.s in $(\mathcal{A}, \mathcal{S})$

↙ ∃ Alternative proof using dg/A_∞-nerve

Corollary Every algebraic triangulated category arises as the homotopy category of a stable ∞-cat.

e.g. $\mathcal{D}(\mathrm{Mod}_R)$

Def / Prop (e.g. Nikolaus - Scholze 2018, also Cisinski 2019)

$\mathcal{C}$: stable ∞-cat & $\mathcal{D} \subseteq \mathcal{C}$: full stable subcat $\rightsquigarrow$ $W_{\mathcal{D}} := \{ f \mid \mathrm{cofib}(f) \in \mathcal{D} \}$

$\Longrightarrow$ $\mathcal{C}/_{\mathcal{D}} := \mathcal{C}[W_{\mathcal{D}}^{-1}]$: stable ∞-cat & $\gamma : \mathcal{C} \longrightarrow \mathcal{C}/_{\mathcal{D}}$ exact

$\forall \mathcal{E}$: stable ∞-cat $\gamma^* : \mathrm{Fun}^{ex}(\mathcal{C}/_{\mathcal{D}}, \mathcal{E}) \xrightarrow{\sim} \mathrm{Fun}^{ex}_{\mathcal{D}}(\mathcal{C}, \mathcal{E})$ equivalence

↑ $\mathcal{D} \ni d \mapsto 0$

Thm $\mathcal{C}_i$, $i \in I$, set-indexed family of stable ∞-cat's $\Rightarrow$ $\prod_{i \in I} \mathcal{C}_i$: stable ∞-cat

$$\begin{tikzcd} \mathcal{C} \times_{\mathcal{E}} \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D} \arrow[r, "G"'] & \mathcal{E} \end{tikzcd}$$
F, G: exact functors between stable ∞-cat's $\Longrightarrow$ $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$: stable ∞-cat

Def (Bernstein - Beilinson - Deligne - Gabber 1982) $\mathcal{C}, \mathcal{D}, \mathcal{E}$: stable ∞-cat's

Recollement :
$$\begin{tikzcd} \mathcal{D} \arrow[r, hook, "i"] & \mathcal{E} \arrow[l, bend right=50, "i_L"'] \arrow[l, bend left=50, "i_R"] \arrow[r, "p"] & \mathcal{C} \arrow[l, bend right=50, "p_L"'] \arrow[l, bend left=50, "p_R"] \end{tikzcd}$$

$p_L \dashv p \dashv p_R$ $\quad i_L \circ i \simeq 1\!\!1_{\mathcal{D}} \simeq i_R \circ i$ $\quad \mathrm{Im}(i) = \mathrm{Ker}(p)$

$i_L \dashv i \dashv i_R$ $\quad p \circ p_L \simeq 1\!\!1_{\mathcal{C}} \simeq p \circ p_L$

$(\Rightarrow \mathcal{E}/_{\mathcal{D}} \xrightarrow{\sim} \mathcal{C}$ & $\mathcal{E}/_{\mathcal{C}} \xrightarrow{\sim} \mathcal{D})$

Gluing $F : \mathcal{C} \longrightarrow \mathcal{D}$ exact functor between stable ∞-cat's

$$\begin{tikzcd} \mathcal{C} \overset{\rightarrow}{\times}_F \mathcal{D} \arrow[r, "p_{\mathcal{C}}"] \arrow[d, "p_{\vec{\mathcal{D}}}"'] \arrow[dr, phantom, "PB"] & \mathcal{C} \arrow[d, "F"] \\ \mathrm{Fun}(s \to t, \mathcal{D}) \arrow[r, "s^*"'] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \overset{\leftarrow}{\times}_F \mathcal{D} \arrow[r, "p_{\mathcal{C}}"] \arrow[d, "p_{\vec{\mathcal{D}}}"'] \arrow[dr, phantom, "PB"] & \mathcal{C} \arrow[d, "F"] \\ \mathrm{Fun}(s \to t, \mathcal{D}) \arrow[r, "t^*"'] & \mathcal{D} \end{tikzcd}$$

stable ∞-cat's

$\mathcal{C} \overset{\rightarrow}{\times}_F \mathcal{D}$: $\{ (c, f : F(c) \to d) \mid c \in \mathcal{C}, f \text{ in } \mathcal{D} \}$

$\mathcal{C} \overset{\leftarrow}{\times}_F \mathcal{D}$: $\{ (c, g : F(c) \leftarrow d) \mid c \in \mathcal{C}, g \text{ in } \mathcal{D} \}$

Recollement

c.f. pull-back of split short exact sequence is split short exact

$$\begin{tikzcd}
\mathcal{D} \arrow[r, hook, "i"] \arrow[d, equal] & \mathcal{C}\overset{\rightarrow}{\underset{F}{\times}}\mathcal{D} \arrow[l, bend right, "i_L"'] \arrow[l, bend left, "i_R"] \arrow[r, "p"] \arrow[d, "p_{\vec{\mathcal{D}}}"] & \mathcal{C} \arrow[l, bend right, "p_L"'] \arrow[l, bend left, "p_R"] \arrow[d, "F"] \\
\mathcal{D} \arrow[r, hook, "0\to ?"'] & \mathrm{Fun}(s\to t, \mathcal{D}) \arrow[l, bend right, "\mathrm{cofib}"'] \arrow[l, bend left, "t^*"] \arrow[r, "s^*"'] & \mathcal{D} \arrow[l, bend right, "?\xrightarrow{1}?"'] \arrow[l, bend left, "?\to 0"]
\end{tikzcd}$$

$i(d) = (0, F(0) \to d)$

$i_R(c, f: F(c) \to d) = d$

$i_L(c, f: F(c) \to d) = \mathrm{cofib}(f)$

$p(c, f: F(c), d) = c$

$p_R(c) = (c, F(c) \to 0)$

$p_L(c) = (c, F(c) \xrightarrow{1} F(c))$

$\boxed{i_R \circ p_L \simeq F}$

Thm

$$\begin{tikzcd}
\mathcal{D} \arrow[r, hook, "i"] & \mathcal{E} \arrow[l, bend right, "i_L"'] \arrow[l, bend left, "i_R"] \arrow[r, "p"] & \mathcal{C} \arrow[l, bend right, "p_L"'] \arrow[l, bend left, "p_R"]
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
\mathcal{E} \arrow[dr, "\sim"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "q"'] & & \\
& \mathcal{C}\overset{\rightarrow}{\underset{F}{\times}}\mathcal{D} \arrow[r, "p_{\mathcal{C}}"] \arrow[d, "p_{\vec{\mathcal{D}}}"'] \arrow[dr, phantom, "PB"] & \mathcal{C} \arrow[d, "F"] \\
& \mathrm{Fun}(s\to t, \mathcal{D}) \arrow[r, "s^*"'] & \mathcal{D}
\end{tikzcd}$$

recollement of stable ∞-cat's

$F := i_R \circ p_L$

$q := i_R(p_L \circ p \xrightarrow{\varepsilon} \mathbb{1})$

Example R, S: rings & M: S-R-bimodule $\rightsquigarrow$ $\begin{pmatrix} S & M \\ 0 & R \end{pmatrix} := \left\{ \begin{pmatrix} s & m \\ 0 & r \end{pmatrix} \;\middle|\; \begin{matrix} s\in S & m \in M \\ & r \in R \end{matrix} \right\}$

$\rightsquigarrow$ recollement:
$$\begin{tikzcd}
\mathcal{D}(R) \arrow[r, hook] & \mathcal{D}\begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \arrow[l, bend right] \arrow[l, bend left, "i_R"] \arrow[r] & \mathcal{D}(S) \arrow[l, bend right, "p_L"'] \arrow[l, bend left]
\end{tikzcd}
\Longrightarrow \quad \mathcal{D}\begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \xrightarrow{\sim} \mathcal{D}(S) \overset{\rightarrow}{\underset{T_M}{\times}} \mathcal{D}(R)$$

with $i_R \circ p_L \simeq - \overset{\mathbb{L}}{\underset{S}{\otimes}} M =: T_M$

Lemma $\mathcal{C}\overset{\leftarrow}{\underset{F}{\times}}\mathcal{D} \xrightarrow{\sim} \mathcal{C}\overset{\rightarrow}{\underset{F}{\times}}\mathcal{D}$, $(c, g: d \to F(c)) \longmapsto (c, F(c) \to \mathrm{cofib}(f))$

Proof (Dyckerhoff - J - Walde 2019)

$$\begin{tikzcd}[column sep=small]
& \mathcal{C}\overset{\leftarrow}{\underset{F}{\times}}\mathcal{D} \arrow[rr] \arrow[dl] \arrow[dd, "\wr"] & & \mathcal{C} \arrow[dl, "F"] \arrow[dd, equal] \\
\mathrm{Fun}(0\to 1, \mathcal{D}) \arrow[rr, "t^*"'] \arrow[dd, "\wr"', "\mathrm{Cofib}"] & & \mathcal{D} \arrow[dd, equal] & \\
& \mathcal{C}\overset{\rightarrow}{\underset{F}{\times}}\mathcal{D} \arrow[rr] \arrow[dl] & & \mathcal{C} \arrow[dl] \\
\mathrm{Fun}(1\to 2, \mathcal{D}) \arrow[rr, "s^*"'] & & \mathcal{D} &
\end{tikzcd}$$

■

Rmk Vast generalisation by Ayala - Mazel-Gee - Rozenblyum (2019, 2023+)

Lemma $L: \mathcal{C} \rightleftarrows \mathcal{D} : R$ adjunction between stable ∞-cat's

$\Longrightarrow$ $\mathcal{C}\overset{\rightarrow}{\underset{L}{\times}}\mathcal{D} \xrightarrow{\sim} \mathcal{D}\overset{\leftarrow}{\underset{R}{\times}}\mathcal{C}$, $(c, f: F(c) \to d) \longmapsto (d, \bar{f}: c \to F(d))$

Thm (Ladkani 2011, Maycock 2011, J 2023+)

rings dg alg's

R, S, E: ring spectra, $- \overset{\mathbb{L}}{\underset{E}{\otimes}} T : \mathcal{D}(E) \xrightarrow{\sim} \mathcal{D}(R)$ equivalence, ${}_S M_R \in \mathcal{D}(S^{op} \otimes R)$ s.t. $M_R \in \mathcal{D}(R)$ is compact

$$\Longrightarrow \mathcal{D}\begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \overset{\sim}{\longleftrightarrow} \mathcal{D}\begin{pmatrix} E & \mathbb{R}\mathrm{Hom}_R(M,T) \\ 0 & S \end{pmatrix}$$

→ includes rings & dg rings (can incorporate k linear structures)

§ t-structures & pre-stable ∞-categories

Abelian cat with small coproducts
+ filtered colim's of s.e.s. are s.e.s
+ $\exists G \in \mathcal{A}$: generator, i.e $\mathrm{Hom}(G,-)$ faithful

Question Universal property of $\mathcal{G} \hookrightarrow \mathcal{D}(\mathcal{G})$, $\mathcal{G}$: Grothendieck cat. ?

e.g. $\mathcal{G} = \mathrm{Mod}_R$

Def (BBDG 1982) $\mathcal{C}$: stable ∞-cat.

aisle coaisle (homological indexing convention!)

A t-structure on $\mathcal{C}$ is a pair $(\underbrace{\mathcal{C}_{\geq 0}}_{\text{aisle}}, \underbrace{\mathcal{C}_{\leq 0}}_{\text{coaisle}})$ of full subcategories of $\mathcal{C}$ s.t.

(1) $\Sigma(\mathcal{C}_{\geq 0}) \subseteq \mathcal{C}_{\geq 0}$, $\Omega(\mathcal{C}_{\leq 0}) \subseteq \mathcal{C}_{\leq 0}$

(2) $\forall X \in \mathcal{C}_{\geq 0}$ $\forall Y \in \mathcal{C}_{\leq -1}$ $\mathrm{Ho}(\mathcal{C})(X,Y) = 0$

(3) $\forall X \in \mathcal{C}$ $\exists$ $\tau_{\geq 0} X \to X \to \tau_{\leq 0} X \to\!\!\!\!+$ triangle in $\mathrm{Ho}(\mathcal{C})$

with $\tau_{\geq 0} X \in \mathcal{C}_{\geq 0}$ & $\tau_{\leq 0} X \in \mathcal{C}_{\leq -1}$

$\mathcal{C}_{\geq n} := \Sigma^n \mathcal{C}_{\geq 0}$

$\mathcal{C}_{\leq n} := \Sigma^n \mathcal{C}_{\leq 0}$

Thm (BBDG 1982) $\mathcal{C}$: stable ∞-cat & $t = (\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$: t-structure.

$$\Longrightarrow \begin{tikzcd} \mathcal{C}_{\geq 0} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "\exists \tau_{\geq 0}"] \arrow[r, bend left, "\exists \tau_{\leq 0}"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{\leq 0} \arrow[l, bend left] \end{tikzcd} ,$$

heart of t

$\mathcal{C}^{\heartsuit} := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$: abelian cat

$\mathrm{Ext}_{\mathcal{C}^{\heartsuit}}(X,Y) \cong \mathrm{Ho}(\mathcal{C})(X, \Sigma Y)$, $X, Y \in \mathcal{C}^{\heartsuit}$

$\pi_k^t := \tau_{\geq k} \circ \tau_{\leq k} : \mathcal{C} \longrightarrow \mathcal{C}^{\heartsuit}$, $\pi_k^t(X) = \pi_0^t(\Omega^k X)$, $k \in \mathbb{Z}$

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \text{ in } \mathcal{C} \rightsquigarrow \cdots \to \pi_1^t(Z) \to \pi_0^t(X) \to \pi_0^t(Y) \to \pi_0^t(Z) \to \pi_{-1}^t(X) \to \cdots \text{ ex. in } \mathcal{C}^{\heartsuit}$$

Rmk $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ & $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ are extension closed subcategories

Example $\mathcal{G}$: Grothendieck cat (e.g. Mod_R)

$\mathcal{D}(\mathcal{G})$ has std. t-structure $\mathcal{D}(\mathcal{G})_{\geq 0} := \{X \in \mathcal{D}(\mathcal{G}) \mid \forall i < 0 \ H_i(X) = 0\}$

$\mathcal{D}(\mathcal{G})_{\leq 0} := \{X \in \mathcal{D}(\mathcal{G}) \mid \forall i > 0 \ H_i(X) = 0\}$

with heart $\mathcal{D}(\mathcal{G})^{\heartsuit} \simeq \mathcal{G}$ & $\pi_i(X) \cong H_i(X)$

homological indexing convention!

Def $\mathcal{C}, \mathcal{D}$: stable ∞-cat's with t-structures $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ & $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$.

An exact functor $F : \mathcal{C} \to \mathcal{D}$ is t-exact if $\underset{\substack{\uparrow \\ \text{right t-exact}}}{F(\mathcal{C}_{\geq 0}) \subseteq \mathcal{D}_{\geq 0}}$ & $\underset{\substack{\uparrow \\ \text{left t-exact}}}{F(\mathcal{C}_{\leq 0}) \subseteq \mathcal{D}_{\leq 0}}$

Non-std def $\to$ $\mathcal{C} \simeq \mathcal{M}[W^{-1}]$: ∞-cat loc. of a combinatorial model cat (Dugger, but see Simpson, Lurie, ...)

Def $\mathcal{C}$: presentable stable ∞-cat. A t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is Grothendieck if

(1) $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ is a presentable ∞-category

(2) $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ is closed under filtered colimits

(3) $\mathcal{C} \xrightarrow{\sim} \lim\left(\cdots \xrightarrow{\Omega} \mathcal{C}_{\geq 0} \xrightarrow{\Omega} \mathcal{C}_{\geq 0} \xrightarrow{\Omega} \mathcal{C}_{\geq 0}\right) =: \mathrm{Sp}(\mathcal{C})$

$\to$ Equivalent to $\mathcal{C}_{\leq \infty} := \bigcap_{n\in\mathbb{Z}} \mathcal{C}_{\leq n} = \{0\}$ since $\mathcal{C}$ admits countable coproducts

Prop $\mathcal{C}$: presentable stable ∞-cat. with Grothendieck t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$

$\Longrightarrow$ $\mathcal{C}^{\heartsuit}$: Grothendieck cat.

Thm $\mathcal{G}$: Grothendieck cat. $\Longrightarrow$ $(\mathcal{D}(\mathcal{G})_{\geq 0}, \mathcal{D}(\mathcal{G})_{\leq 0})$ is a Grothendieck t-str. on $\mathcal{D}(\mathcal{G})$

& $\mathcal{D}(\mathcal{G})_{\geq \infty} := \bigcap_{n\in\mathbb{Z}} \mathcal{D}(\mathcal{G})_{\geq n} = \{0\}$

$\forall$ $\mathcal{C}$: presentable stable ∞-cat. with Grothendieck t-str. $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$

such that $\boxed{\mathcal{C}_{\geq \infty} = \{0\}}$ ⊛, restriction to the heart induces an equivalence

t-exact functors $\qquad$ exact functors

$$\mathrm{LFun}^{t\text{-ex}}(\mathcal{D}(\mathcal{G}), \mathcal{C}) \xrightarrow{\sim} \mathrm{LFun}^{ex}(\mathcal{G}, \mathcal{C}^{\heartsuit})$$

colimit-preserving

Realisation functors $\mathcal{C}$: presentable stable ∞-cat. with Grothendieck t-str. $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ + ⊛

$$\begin{array}{ccc} \mathrm{LFun}^{t\text{-ex}}(\mathcal{D}(\mathcal{C}^{\heartsuit}), \mathcal{C}) & \xrightarrow{\sim} & \mathrm{LFun}^{ex}(\mathcal{C}^{\heartsuit}, \mathcal{C}^{\heartsuit}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathrm{Real}_t & \longmapsto & \mathbb{1}_{\mathcal{C}^{\heartsuit}} \end{array}$$

$$\begin{tikzcd} \mathcal{D}(\mathcal{C}^{\heartsuit}) \arrow[r, "\mathrm{Real}_t"] & \mathcal{C} \\ \mathcal{C}^{\heartsuit} \arrow[u, hook] \arrow[r, "\mathbb{1}_{\mathcal{C}^{\heartsuit}}"'] & \mathcal{C}^{\heartsuit} \arrow[u, hook] \end{tikzcd}$$

Can be constructed by other means, e.g. filtered derived cat's, derivators...

Def $\mathcal{C}$: ∞-cat is pre-stable if

(0) $\exists\, 0 \in \mathcal{C}$: zero object and $\mathcal{C}$ admits finite colimits

(1) $\Sigma : \mathcal{C} \to \mathcal{C}$ is fully faithful

(2) $\forall f : Y \to \Sigma Z$ in $\mathcal{C}$ there exists
$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ 0 \arrow[r] & \Sigma Z \end{tikzcd}$$
bicartesian in $\mathcal{C}$

Thm $\mathcal{C}$: ∞-cat with zero object & finite colimits. TFAE

(a) $\mathcal{C}$ is prestable & admits finite limits

(b) $\mathcal{C}$ is equivalent to an extension-closed full subcat of some stable ∞-cat $\mathcal{D}$

that is moreover closed under finite colimits / the aisle of a t-structure on $\mathcal{D}$

DISCLAIMER Unless noted otherwise, unattributed results are due to Lurie (at least in the form we present)

§ The universal stable ∞-category

$$\begin{tikzcd} \mathcal{A}\colon \text{abelian category} \arrow[r, squiggly] & \mathcal{A}(-,-)\colon \mathcal{A}^{op}\times\mathcal{A} \arrow[r] & \textcircled{\mathrm{Ab}} \arrow[d, squiggly] \\ \text{Question } \mathcal{C}\colon \text{stable } \infty\text{-cat} \arrow[r, squiggly] & \underline{\mathrm{Map}}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{op}\times\mathcal{C} \arrow[r] & \textcircled{?} \end{tikzcd}$$

$(\mathrm{Gpd}_\infty)_*$:= ∞-cat of pointed ∞-groupoids $(X, x) = (* \xrightarrow{x} X)$

Def The ∞-cat of spectra is $\mathrm{Sp} := \lim\left(\cdots \xrightarrow{\Omega} (\mathrm{Gpd}_\infty)_* \xrightarrow{\Omega} (\mathrm{Gpd}_\infty)_* \xrightarrow{\Omega} (\mathrm{Gpd}_\infty)_*\right)$

Thm TFSH

(1) Sp : presentable stable ∞-cat

(2) $\exists\ \Sigma^\infty_+ : \mathrm{Gpd}_\infty \rightleftarrows \mathrm{Sp} : \Omega^\infty$ adjunction

Free spectrum — Underlying ∞-groupoid

(3) $\mathbb{S} := \Sigma^\infty_+(*)$: sphere spectrum is a compact generator:

- $\mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}, -) : \mathrm{Ho}(\mathrm{Sp}) \longrightarrow \mathrm{Ab}$ preserves small coproducts
- $\forall X \in \mathrm{Sp}\ \left(\forall i \in \mathbb{Z}\ \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^i(\mathbb{S}), X) = 0 \Rightarrow X = 0\right)$

Moreover, $\mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}, \Sigma^{>0}\mathbb{S}) = 0$

By definition $\mathbb{S} \in \mathrm{Sp}$ is a compact silting object

(4) Write $\pi_i := \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^i\mathbb{S}, -) : \mathrm{Ho}(\mathrm{Sp}) \longrightarrow \mathrm{Ab}$

The following pair $(\mathrm{Sp}_{\geq 0}, \mathrm{Sp}_{\leq 0})$ is a Grothendieck t-structure on Sp with heart $\mathrm{Sp}^\heartsuit \simeq \mathrm{Ab}$ and such that $\mathrm{Sp}_{\geq\infty} = \{0\} = \mathrm{Sp}_{\leq\infty}$

$\mathbb{S} \in \mathrm{Sp}_{\geq 0} := \{X \in \mathrm{Sp} \mid \forall i < 0\ \pi_i(X) = 0\}$ $\qquad$ $\mathrm{Sp}_{\leq 0} := \{X \in \mathrm{Sp} \mid \forall i > 0\ \pi_i(X) = 0\}$

(5) $\forall \mathcal{C}$: presentable stable ∞-cat. $\mathrm{ev}_{\mathbb{S}} : \mathrm{LFun}(\mathrm{Sp}, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}$ is an equivalence

(6) $\forall \mathcal{C}$: stable ∞-cat $\exists\ \underline{\mathrm{Map}}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C}^{op} \longrightarrow \mathrm{Sp}$ s.t. $\tau_{\geq 0}\underline{\mathrm{Map}}_{\mathcal{C}} \simeq \mathrm{Map}_{\mathcal{C}}$

§ Exact ∞-categories

$(\mathcal{A}, \mathcal{S})$: exact 1-cat $\rightsquigarrow$ $\mathcal{A} \underset{\text{ext-closed}}{\hookrightarrow} \mathcal{D}^b(\mathcal{A}, \mathcal{S})$

stable ∞-cat's

$\mathcal{C}$: pre-stable ∞-cat $\rightsquigarrow$ $\mathcal{C} \underset{\text{ext-closed}}{\hookrightarrow} SW(\mathcal{C}) = \operatorname{colim}(\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \cdots)$

Aim Axiomatise extension-closed subcategories of stable ∞-cat's and relate these to Nakaoka-Palu extriangulated cat's.

Def An ∞-category $\mathcal{A}$ is <u>additive</u> if

(0) $\exists\, 0 \in \mathcal{A}$: zero object

(1) $\forall X, Y \in \mathcal{A}$ $\exists\, X \amalg Y$, $X \times Y \in \mathcal{A}$

(2) $Ho(\mathcal{A})$ is additive, i.e

- $\forall X, Y \in \mathcal{A}$ $\quad X \amalg Y \xrightarrow{\begin{pmatrix}1&0\\0&1\end{pmatrix}} X \times Y$ is an isomorphism
- $\forall X \in \mathcal{A}$ $\quad X \oplus X \xrightarrow{\begin{pmatrix}1&1\\0&1\end{pmatrix}} X \oplus Y$ is invertible

Rmk Being additive is a property of $\mathcal{A}$

Example Every additive cat is additive when viewed as an ∞-cat.

Prove first, then use $\mathcal{C} \hookrightarrow SW(\mathcal{C})$

Example $\mathcal{C}$: stable ∞-cat $\Longrightarrow$ $\mathcal{C}$: pre-stable ∞-cat. $\Rightarrow$ $\mathcal{C}$: additive ∞-cat.

Thm $\mathcal{A}$: small additive ∞-cat. TFSH

finite-product preserving

(1) $\mathcal{A} \xrightarrow{y} \operatorname{Fun}^{\pi}(\mathcal{A}^{op}, Grp_\infty)$, $X \longmapsto \operatorname{Map}_{\mathcal{A}}(-, X)$, is fully faithful

pre-stable ∞-cat

(2) $\Omega^\infty : Sp_{\geq 0} \to Gpd_\infty$ induces an equivalence $\boxed{\operatorname{Fun}^{\pi}(\mathcal{A}^{op}, Sp_{\geq 0})} \xrightarrow{\sim} \operatorname{Fun}^{\pi}(\mathcal{A}^{op}, Gpd_\infty)$

Corollary $\mathcal{A}$: additive ∞-cat $\rightsquigarrow$ $\underline{\operatorname{Map}}_{\mathcal{A}}(-,-)_{\geq 0} : \mathcal{A}^{op} \times \mathcal{A}^{op} \longrightarrow Sp_{\geq 0}$

not Barwick's original def. Equivalence proven e.g. in upcoming work: J-Kvamme-Palu-Walde

Def (Quillen 1972, Barwick 2015) $\mathcal{A}$: additive ∞-cat.

$\mathcal{S}$: class of bicartesian squares in $\mathcal{A}$ of the form
$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & Y \arrow[d, two heads, "p"] \\ 0 \arrow[r, tail] & Z \end{tikzcd}$$

We call $\mathcal{S}$ an exact structure on $\mathcal{A}$ if the following axioms are satisfied:

(Ex0) $\mathcal{S}$ is closed under isomorphisms & $\forall X \in \mathcal{A}$
$$\begin{tikzcd} X \arrow[r, tail, "1"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ 0 \arrow[r, tail] & 0 \end{tikzcd}, \quad \begin{tikzcd} 0 \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & X \arrow[d, two heads, "1"] \\ 0 \arrow[r, tail] & X \end{tikzcd} \in \mathcal{S}$$

\+

(Ex1)
$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & Y \arrow[d, two heads, "p"] \\ 0 \arrow[r, tail] & W \end{tikzcd}, \quad \begin{tikzcd} Y \arrow[r, tail, "j"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & Z \arrow[d, two heads, "q"] \\ 0 \arrow[r, tail] & W' \end{tikzcd} \in \mathcal{S} \Rightarrow \exists \begin{tikzcd} X \arrow[r, tail, "k"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & Y \arrow[d, two heads, "r"] \\ 0 \arrow[r, tail] & W'' \end{tikzcd} \in \mathcal{S} \text{ with } \begin{tikzcd} & Y \arrow[dr, tail, "j"] & \\ X \arrow[ur, tail, "i"] \arrow[rr, "k"'] & & Y \end{tikzcd}$$

\+

(Ex2)
$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & Y \arrow[d, two heads, "p"] \\ 0 \arrow[r, tail] & W \end{tikzcd} \in \mathcal{S} \;\&\; f: X \to X' \Rightarrow \exists \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail, "i"] \arrow[dd] & & X' \arrow[dr, tail, "i"] \arrow[dd] & \\ & Y \arrow[rr] \arrow[dd, two heads, "p"] & & Y' \arrow[dd, two heads] \\ 0 \arrow[rr] \arrow[dr] & & 0 \arrow[dr] & \\ & Z \arrow[rr] & & Z' \end{tikzcd} \text{ in } \mathcal{A}.$$

\+

(Ex1^{op}) + (Ex2^{op}) $\rightsquigarrow$ $(\mathcal{A}, \mathcal{S})$: exact ∞-cat.

Rmk (X. Chen 2023) Variant in the context of dg categories.

Example $(\mathcal{A}, \mathcal{S})$: exact 1-cat $\Rightarrow$ $(\mathcal{A}, \mathcal{S})$: exact ∞-cat

Example $\mathcal{A}$: additive ∞-cat, $\mathcal{S}_\oplus := \{$split fibre-cofibre sequences$\}$
$\Rightarrow$ $(\mathcal{A}, \mathcal{S}_\oplus)$: exact ∞-cat

Example $(\mathcal{A}, \mathcal{S})$: exact ∞-cat & $\mathcal{B} \subseteq \mathcal{A}$: full additive subcategory
$\mathcal{B}$: closed under $\mathcal{S}$-extensions $\Rightarrow$ $(\mathcal{B}, \mathcal{S}|_{\mathcal{B}})$: exact ∞-cat

Example $\mathcal{C}$: stable ∞-cat, $\mathcal{S}_{max}$: all fibre-cofibre sequences in $\mathcal{C}$
$\Rightarrow$ $(\mathcal{C}, \mathcal{S}_{max})$: exact ∞-cat

Example $\mathcal{C}$: pre-stable ∞-cat, $\mathcal{S}_{max}$: all fibre-cofibre sequences in $\mathcal{C}$
$\Rightarrow$ $(\mathcal{C}, \mathcal{S}_{max})$: exact ∞-cat

Def $(\mathcal{A}, \mathcal{S}_{\mathcal{A}}), (\mathcal{B}, \mathcal{S}_{\mathcal{B}})$: exact ∞-cat's. $F : \mathcal{A} \to \mathcal{B}$ is exact if $F(0) \simeq 0$

$$\begin{tikzcd}
 & X \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & Y \arrow[d, two heads] & & F(X) \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & F(Y) \arrow[d] & \\
\forall & 0 \arrow[r, tail] & Z & \in \mathcal{S}_{\mathcal{A}} \arrow[r, mapsto, "F"] & F(0) \arrow[r, tail] & F(Z) & \in \mathcal{S}_{\mathcal{B}}
\end{tikzcd}$$

Def Ex_∞ : ∞-cat of (ess. small) exact ∞-cat's & exact functors $\qquad (\mathcal{C}, \mathcal{S}_{max})$

$$\begin{tikzcd}
\mathrm{Ex}_\infty & (\mathcal{C}, \mathcal{S}_{max}) \\
\mathrm{St}_\infty \arrow[u, hook, "\iota"'] & \mathcal{C} \arrow[u, mapsto]
\end{tikzcd}$$

St_∞ : ∞-cat of (ess. small) stable ∞-cat's & exact functors

Thm (Klemenc 2022) TFSH

(1) $\exists\ \overset{\text{Stable hull}}{\mathcal{H}_{st}} : \mathrm{Ex}_\infty \rightleftarrows \mathrm{St}_\infty : \iota$ adjunction, $\overset{\text{unit}}{\eta} : (\mathcal{A}, \mathcal{S}) \underset{\text{exact}}{\longrightarrow} (\mathcal{H}_{st}(\mathcal{A}, \mathcal{S}), \mathcal{S}_{max})$

$\forall\, \mathcal{C}$: stable ∞-cat $\quad \eta^* : \mathrm{Fun}^{ex}(\mathcal{H}_{st}(\mathcal{A}, \mathcal{S}), \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}^{ex}((\mathcal{A}, \mathcal{S}), (\mathcal{C}, \mathcal{S}_{max}))$

(2) $\eta : \mathcal{A} \longrightarrow \mathcal{H}_{st}(\mathcal{A}, \mathcal{S})$ is fully faithful

- $\eta(\mathcal{A}) \subseteq \mathcal{H}_{st}(\mathcal{A}, \mathcal{S})$ is closed under extensions
- $$\begin{tikzcd}
\eta(X) \arrow[r, "\eta(i)"] \arrow[d] \arrow[dr, phantom, "\square"] & \eta(Y) \arrow[d, "\eta(p)"] & & X \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "p"] & \\
0 = \eta(0) \arrow[r] & \eta(Z) & \text{in } \mathcal{H}_{st}(\mathcal{A}) \ \Longrightarrow & 0 \arrow[r] & Z & \in \mathcal{S}
\end{tikzcd}$$

Corollary (cf. Børve – Trygsland 2021)

$(\mathcal{A}, \mathcal{S})$: exact ∞-cat $\Rightarrow \exists\ \underline{\mathrm{Map}}_{\mathcal{S}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathrm{Sp}$

$\pi_{<0} \underline{\mathrm{Map}}_{\mathcal{S}}(-,-) =: \mathrm{Ext}^{>0}_{\mathcal{S}}(-,-)$

Corollary (Nakaoka – Palu 2020)

$(\mathcal{A}, \mathcal{S})$: exact ∞-cat $\Rightarrow (\mathrm{Ho}(\mathcal{A}), \pi_0 \underline{\mathrm{Map}}_{\mathcal{S}}(-, \Sigma(-)), \mathcal{S})$: extri. cat

Thm (Bunke – Cisinski – Kasprowski – Winges 2019)

$(\mathcal{A}, \mathcal{S})$: exact 1-cat $\Rightarrow \mathcal{H}_{st}(\mathcal{A}, \mathcal{S}) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{A}, \mathcal{S})$

Example (Lurie) $\mathcal{C}$: pre-stable ∞-cat $\Rightarrow \mathcal{H}_{st}(\mathcal{C}, \mathcal{S}_{max}) \xrightarrow{\sim} \mathrm{SW}(\mathcal{C})$

§ Construction of the stable hull, after Klemenc

$\mathcal{A}$: additive ∞-cat.

preserves finite coproducts　　　　　　　　preserve finite products

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook, "y"] \arrow[dr, hook] \arrow[ddr, "\tilde{\eta}"'] & \mathcal{P}_\Sigma(\mathcal{A}) := \mathrm{Fun}^\pi(\mathcal{A}^{op}, \mathrm{Gpd}_\infty) & \mathrm{Fun}^\pi(\mathcal{A}^{op}, \mathrm{Sp}_{\geq 0}) \arrow[l, "\sim"'] \\
& \mathcal{P}^f_\Sigma(\mathcal{A}) \arrow[u, phantom, "\cup"] \arrow[d, hook] & \\
& SW\left(\mathcal{P}^f_\Sigma(\mathcal{A})\right) &
\end{tikzcd}$$

$\mathcal{P}^f_\Sigma(\mathcal{A})$: full subcategory generated by $y(\mathcal{A})$ under finite colimits

$$SW\left(\mathcal{P}^f_\Sigma(\mathcal{A})\right) := \operatorname{colim}\left(\mathcal{P}^f_\Sigma(\mathcal{A}) \xrightarrow{\Sigma} \mathcal{P}^f_\Sigma(\mathcal{A}) \xrightarrow{\Sigma} \mathcal{P}^f_\Sigma(\mathcal{A}) \xrightarrow{\Sigma} \dots\right)$$

e.g. R: ring $\rightsquigarrow$ $\mathcal{P}_\Sigma(\mathrm{free}_R) \xrightarrow{\sim} \mathcal{D}(\mathrm{Mod}_R)_{\geq 0}$ standard aisle

Prop (Klemenc 2022, Lurie) $\mathcal{A}$: additive ∞-cat

preserve finite products

$$\forall \mathcal{C}: \text{stable } \infty\text{-cat} \quad \tilde{\eta}^* : \mathrm{Fun}^{ex}\left(SW\left(\mathcal{P}^f_\Sigma(\mathcal{A})\right), \mathcal{C}\right) \xrightarrow{\sim} \mathrm{Fun}^\pi(\mathcal{A}, \mathcal{C})$$

Example $\mathcal{A}$: additive 1-cat $\Longrightarrow$ $SW\left(\mathcal{P}^f_\Sigma(\mathcal{A})\right) \simeq \mathcal{D}^b(\mathcal{A}, \mathcal{S}_\oplus) = \mathcal{K}^b(\mathcal{A})$

$(\mathcal{A}, \mathcal{S})$: exact ∞-cat.

Problem $\tilde{\eta}: \mathcal{A} \hookrightarrow SW\left(\mathcal{P}^f_\Sigma(\mathcal{A})\right)$, $X \longmapsto \hat{X}$, does not preserve finite colimits

Construction

$$\begin{tikzcd}
X \arrow[r, tail, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "p"] \\
0 \arrow[r, tail] & Z
\end{tikzcd} \in \mathcal{S} \rightsquigarrow \underset{\underset{SW\left(\mathcal{P}^f_\Sigma(\mathcal{A})\right)}{\Cap}}{\mathrm{tot}(S)} := \operatorname{colim}\left(\begin{tikzcd}[row sep=small, column sep=small]
\hat{X} \arrow[rr, "\hat{i}"] \arrow[dr] \arrow[dd] & & \hat{Y} \arrow[dr, "\hat{p}"] \arrow[dd] & \\
& 0 \arrow[rr] \arrow[dd] & & \hat{Z} \arrow[dl] \\
0 \arrow[rr] \arrow[dr] & & 0 & \\
& 0 & &
\end{tikzcd}\right)$$

ⓢ

not bicartesian in $SW(\mathcal{P}^f_\Sigma(\mathcal{A}))$ in general

Example $(\mathcal{A}, \mathcal{S})$: additive 1-cat $\rightsquigarrow$ $SW\left(\mathcal{P}^f_\Sigma(\mathcal{A})\right) \simeq \mathcal{K}^b(\mathcal{A})$

$$S: X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z \in \mathcal{S} \rightsquigarrow \mathrm{tot}(S) = \left(\cdots \to 0 \to X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z \to 0 \to \cdots\right) \in \mathcal{K}^b(\mathcal{A})$$

$$\boxed{\mathcal{H}_{st}(\mathcal{A}, \mathcal{S}) := SW\left(\mathcal{P}^f_\Sigma(\mathcal{A})\right) / \operatorname{thick}\{\mathrm{tot}(S) \mid S \in \mathcal{S}\}}$$