The Donovan-Wemyss Conjecture via the Derived Auslander-Iyama Correspondence

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§ Contraction algebras & the Donovan-Wemyss Conjecture

Fix R: compound Du Val (cDV) singularity (Reid 1983)

Assume R: isolated singularity & $\exists\, p : X \to \operatorname{Spec} R$ crepant resolution (also fixed)

$\rightsquigarrow$ $\mathcal{O}_X \oplus \mathcal{N} \in \operatorname{coh} X$: Van den Bergh's tilting bundle (2004)

$$\begin{array}{ccccc} D^b(\operatorname{coh} X) & \xrightarrow{p_*} & D^b(\operatorname{mod} R) & \twoheadrightarrow & D^b(\operatorname{mod} R)/K^b(\operatorname{proj} R) =: D_{sing}(R) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathcal{O}_X \oplus \mathcal{N} & \longmapsto & R \oplus N & \longmapsto & T = T(p) \end{array}$$

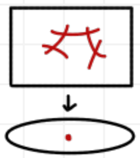
Def (Donovan-Wemyss 2016) The contraction algebra is $\Lambda = \Lambda(p) := \operatorname{End}(T)$

- R: complete local + isolated $\Rightarrow$ $D_{sing}(R)$: Hom-finite & Krull-Remak-Schmidt
- (Auslander 1978) $\dim R = 3 \Rightarrow D_{sing}(R)$: 2-CY triangulated category
- (Eisenbud 1980) R: hypersurface $\Rightarrow [2] \cong \mathbb{1}$ as exact functors
- $\forall\, x \in D_{sing}(R)$ the algebra $\operatorname{End}(x)$ is symmetric:

$$\operatorname{Hom}(x,x) \cong D\operatorname{Hom}(x, x[2]) \cong D\operatorname{Hom}(x,x)$$

$\rightsquigarrow$ Λ is a fin.-dim. basic symmetric algebra

$K_0(\Lambda) \cong \mathbb{Z}^{\ell}$, $\quad p^{-1}(\mathfrak{m}) = \bigcup_{i=1}^{\ell} C_i$, $\quad C_i^{red} \cong \mathbb{P}^1$



$$\begin{array}{c} X \\ \downarrow p \\ \operatorname{Spec} R \end{array}$$

Example (Atiyah flop) $R = \mathbb{C}[\![u,v,x,y]\!]/(uv - xy)$

$$\begin{array}{ccc} X & \not\cong & Y \\ {}_p\searrow & & \swarrow_q \\ & \operatorname{Spec} R & \end{array}$$

$\boxed{\Lambda(p) \cong \mathbb{C} \cong \Lambda(q)}$ $\leftarrow$ Misleadingly simple!

Donovan-Wemyss Conjecture (2016, August 2020)

$p_i : X_i \longrightarrow \operatorname{Spec} R_i$, $i = 1,2$, crepant resolutions of isolated cDV singularities

$$\boxed{R_1 \cong R_2 \text{ as algebras} \iff D^b(\operatorname{mod} \Lambda(p_1)) \simeq D^b(\operatorname{mod} \Lambda(p_2)) \text{ as tri. cat's}}$$

($\Rightarrow$) Type $\mathbb{A}$ (Reid 1982), in general (Wemyss 2018 + Dugas 2015)

($\Leftarrow$) TODAY

Def (Iyama - Yoshino 2008) $T \in D_{sing}(R)$ is 2$\mathbb{Z}$-cluster tilting if $T[2] \cong T$ (automatic)

$$\operatorname{Hom}(T, T[1]) = 0 \quad \& \quad \forall\, X \in D_{sing}(R) \ \exists\, \underbrace{T_1 \to T_0 \to X \to T_1[1]}_{\text{exact triangle}}, \quad T_i \in \operatorname{add}(T)$$

Thm (Wemyss 2018) The map $p \longmapsto T(p)$ yields a bijection

$$\begin{array}{c} \text{flop} \circlearrowright \{\, p : X \to \operatorname{Spec} R \ \text{crepant resolution} \,\}/_{\cong} \\ \updownarrow {\scriptstyle 1:1} \\ \text{mutation} \circlearrowright \{\, T \in D_{sing}(R) : \text{basic } 2\mathbb{Z}\text{-cluster tilting} \,\}/_{\cong} \end{array}$$

Thm (August 2020) Γ : basic fin.-dim. algebra TFAE

(1) $\exists\, p : X \to \operatorname{Spec} R$ crepant resolution s.t. $D^b(\operatorname{mod} \Gamma) \simeq D^b(\operatorname{mod} \Lambda(p))$

(2) $\exists\, q : X \to \operatorname{Spec} R$ crepant resolution s.t. $\Gamma \cong \Lambda(q)$

Application to the DW Conjecture

$p_i : X_i \longrightarrow \operatorname{Spec} R_i$, $i = 1,2$, crepant resolutions of isolated cDV singularities

$D^b(\operatorname{mod} \Lambda(p_1)) \simeq D^b(\operatorname{mod} \Lambda(p_2))$ as tri. cat's

(August 2020) $\exists\, q : X_2 \to \operatorname{Spec} R_2$ crepant resolution s.t. $\Lambda(p_1) \cong \Lambda(q)$

WLOG We may and we will assume that $\Lambda(p_1) \cong \Lambda(p_2)$

§ The Derived Donovan-Wemyss Conjecture

$D_{sing}(R)_{dg} := D^b(\mathrm{mod}\, R)_{dg} \,/\, K^b(\mathrm{proj}\, R)_{dg}$: dg-category of singularities

$\rotatebox{90}{$\in$}$

$T = T(p) \rightsquigarrow \boldsymbol{\Lambda} = \boldsymbol{\Lambda}(p) := \mathbb{R}\mathrm{End}(T)$: $2\mathbb{Z}$-derived contraction algebra

$\Lambda^\bullet = \Lambda^\bullet(p) := H^*(\boldsymbol{\Lambda}) \cong \Lambda[u^\pm] = \Lambda \otimes_{\mathbb{C}} \mathbb{C}[u^\pm]$, $|u^\pm| = -2$

$T \in D_{sing}(R)$: $2\mathbb{Z}$-CT $\Rightarrow$ thick $T = D_{sing}(R)$

(Keller 1994) $\underset{\text{compact/perfect derived cat.}}{D^c}(\boldsymbol{\Lambda})_{dg} \xrightarrow{\sim} D_{sing}(R)_{dg}$ quasi-equivalence of dg categories

Thm (Hua-Keller 2018, 2023+) The 0-th Hochschild cohomology of $D_{sing}(R)_{dg}$ is

$HH^0(D_{sing}(R)_{dg}) \cong$ Tyurina algebra of f, where $R = \mathbb{C}[[x,y,z,t]]/(f)$

(Mather-Yau 1982) $\dim R = 3$ + Tyurina alg. of f determine R up to isomorphism

Corollary (Hua-Keller 2018, 2023+)

$p_i : X_i \longrightarrow \mathrm{Spec}\, R_i$, $i = 1,2$, crepant resolutions of isolated cDV singularities

$$\boxed{R_1 \cong R_2 \text{ as algebras } \Longleftarrow \boldsymbol{\Lambda}(p_1) \simeq \boldsymbol{\Lambda}(p_2) \text{ quasi-iso as dg algebras}}$$

Derived DW Conjecture

Corollary The DW conjecture holds if and only if the following statement holds:

$p_i : X_i \longrightarrow \mathrm{Spec}\, R_i$, $i = 1,2$, crepant resolutions of isolated cDV singularities

$\Lambda(p_1) \cong \Lambda(p_2)$ as algebras $\Longleftrightarrow$ $\boldsymbol{\Lambda}(p_1) \simeq \boldsymbol{\Lambda}(p_2)$ quasi-iso as dg algebras

Warning A, B: dg algebras $H^*(A) \cong H^*(B) \not\Rightarrow A \simeq B$ quasi-iso as dg algebras

However $H^*(A) \cong \mathbb{C}[u^\pm] \Longrightarrow A \simeq \mathbb{C}[u^\pm]$ quasi-iso as dg algebras

Strategy $H^*(A) \cong \Lambda^\bullet$ + (?) $\Longrightarrow A \simeq \boldsymbol{\Lambda}(p)$ quasi-iso as dg algebras

§ The restricted Universal Massey Product

(Kadeishvili 1982) $\Lambda^\bullet = \Lambda^\bullet(p)$ inherits from $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(p)$ a minimal A_∞-alg. structure

$$m_n : \underbrace{\Lambda^\bullet \otimes \cdots \otimes \Lambda^\bullet}_{n \text{ factors}} \longrightarrow \Lambda^\bullet[2-n], \quad n \geq 2$$

such that $\boldsymbol{\Lambda} \simeq \Lambda^\bullet$ quasi-iso as A_∞-algebras.

$\Lambda^\bullet$: concentrated in even degrees $\Longrightarrow \forall n \notin 2\mathbb{Z} \quad m_n = 0$

By definition, $m_4 \in \underset{\text{as graded algebra}}{C^{4,-2}}(\Lambda^\bullet, \Lambda^\bullet)$ is a Hochschild cochain

A_∞-equations $\Longrightarrow \partial_{Hoch}(m_4) = 0 \rightsquigarrow \{m_4\} \in HH^{4,-2}(\Lambda^\bullet, \Lambda^\bullet)$

Def $\{m_4\} \in HH^{4,-2}(\Lambda^\bullet, \Lambda^\bullet)$ is the Universal Massey Product

$j : \Lambda \overset{\deg 0}{\hookrightarrow} \Lambda^\bullet \rightsquigarrow j^* : HH^{\bullet,*}(\Lambda^\bullet, \Lambda^\bullet) \longrightarrow HH^{\bullet,*}(\Lambda, \Lambda^\bullet)$

Def $j^*\{m_4\} \in HH^{4,-2}(\Lambda, \Lambda^\bullet)$ is the restricted Universal Massey Product

Notice $HH^{4,-2}(\Lambda, \Lambda^\bullet) \cong HH^4(\Lambda, \Lambda^{-2}) = HH^4(\Lambda, \Lambda)$

$\rightsquigarrow j^*\{m_4\} \in HH^4(\Lambda, \Lambda) = \mathrm{Ext}^4_{\Lambda^e}(\Lambda, \Lambda)$, $\Lambda^e := \Lambda \otimes \Lambda^{op}$

$\underline{\mathrm{mod}}\, \Lambda^e$: stable cat. of Λ-bimodules $\rightsquigarrow \mathrm{Ext}^4_{\Lambda^e}(\Lambda, \Lambda) \cong \underline{\mathrm{Hom}}_{\Lambda^e}(\Omega^4_{\Lambda^e}(\Lambda), \Lambda)$

Thm (J-Muro 2022) $j^*\{m_4\} : \Omega^4_{\Lambda^e}(\Lambda) \longrightarrow \Lambda$ is an isomorphism in $\underline{\mathrm{mod}}\, \Lambda^e$

The proof is thechical: the claim is equivalent to $T \in D_{sing}(R)$: $2\mathbb{Z}$-cluster tilting

Key point $D_{sing}(R) \supseteq \mathrm{add}(T) \circlearrowleft [\pm 2]$

$$\square = \left\{ \begin{tikzcd}[column sep=small] & T_3 \arrow[rr] \arrow[dr] \arrow[d, phantom, "\Delta"] & & T_2 \arrow[dr] \arrow[d, phantom, "\Delta"] & \\ T_4 \arrow[ur] & {} & * \arrow[ll, "+"] \arrow[ur] \arrow[u, phantom, "="] & {} & T_1 \arrow[ll, "+"] \end{tikzcd} \;\middle|\; T_i \in \mathrm{add}(T) \right\}$$

$(\mathrm{add}(T), [2], \square)$ is a 4-angulated cat (Geiß-Keller-Oppermann 2013)

$j^*\{m_4\}$ detects the 4-angles in the class $\square$ through tensor products

§ Hochschild-Tate cohomology & the main thm

$\bar{\delta} : \Lambda^\bullet \longrightarrow \Lambda^\bullet$, $x \longmapsto \frac{|x|}{2} x$ is the fractional Euler derivation

$\rightsquigarrow \delta := \{\bar{\delta}\} \in HH^{1,0}(\Lambda^\bullet, \Lambda^\bullet)$ the fractional Euler class

$\underline{HH}^{\bullet,*}(\Lambda, \Lambda^\bullet) := \underline{Ext}^{\bullet,*}_{\Lambda^e}(\Lambda, \Lambda^\bullet)$: Hochschild-Tate Cohomology

$HH^{>0,*}(\Lambda, \Lambda^\bullet) = \underline{HH}^{>0,*}(\Lambda, \Lambda^\bullet)$ & $HH^{0,*}(\Lambda, \Lambda^\bullet) \twoheadrightarrow \underline{HH}^{0,*}(\Lambda, \Lambda^\bullet)$

Thm (J-Muro 2022, J-Keller-Muro 2023) There are isomorphisms

$$\begin{tikzcd} HH^{\bullet,*}(\Lambda^\bullet, \Lambda^\bullet) \arrow[r, "\sim"] \arrow[d, "j^*"'] & HH^\bullet(\Lambda,\Lambda)[z^\pm, \delta] \arrow[d, "\delta \mapsto 0"] \\ HH^{\bullet,*}(\Lambda, \Lambda^\bullet) \arrow[r, "\sim"] \arrow[d] & HH^\bullet(\Lambda,\Lambda)[z^\pm] \arrow[d] \\ \underline{HH}^{\bullet,*}(\Lambda, \Lambda^\bullet) \arrow[r, "\sim"] & \underline{HH}^\bullet(\Lambda,\Lambda)[z^\pm] \end{tikzcd}$$

$|z| = (0,-2)$, $[\delta, z] = -z$

as bigraded algebras

$$\boxed{\begin{array}{c} \{m_4\} = u \cdot z \in HH^\bullet(\Lambda,\Lambda)[z^\pm, \delta] \text{ where} \\ u \in HH^4(\Lambda,\Lambda) \cong \underline{Hom}_{\Lambda^e}(\Omega^4_{\Lambda^e}(\Lambda), \Lambda) \text{ is invertible } \& \ [u,u] = 0 \end{array}}$$

Thm (J-Muro 2022)

Up to A_∞-isomorphism, there is a unique minimal A_∞-alg. structure on $\Lambda^\bullet$ s.t.

$$j^*\{m_4\} \in \underline{HH}^{4,-2}(\Lambda, \Lambda^\bullet) \cong \underline{Hom}_{\Lambda^e}(\Omega^4_{\Lambda^e}(\Lambda), \Lambda)$$

is an isomorphism.

Rmk A proof by direct computation is possible (J-Muro-Keller 2023)

Corollary The DW Conjecture holds

§ A perspective from homotopy theory

endomorphism dg operad of $\Lambda^\bullet$

$X_n := Map_{dgOp}(\mathcal{A}_{n+2}, End(\Lambda^\bullet))$: space of minimal A_n-alg structures on $\Lambda^\bullet$

A_{n+2}-operad, $\infty \geq n \geq 0$

$$\boxed{\pi_0(X_\infty) = \{\text{min. } A_\infty\text{-algebra structures on } \Lambda^\bullet\} / A_\infty\text{-isomorphism}}$$

$$X_\infty \xrightarrow{\sim} \underleftarrow{holim}(\cdots \to X_n \to \cdots \to X_1 \to X_0)$$

$x_\infty \in X_\infty$: minimal A_∞-algebra structure on $\Lambda^\bullet$ induced by $\mathcal{A}$.

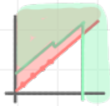
forget m_k, $k > n$

$x_n \in X_n$: minimal A_{n+2}-algebra structure on $\Lambda^\bullet$

$y_\infty \in X_\infty$: minimal A_∞-algebra structure on $\Lambda^\bullet$ s.t. $j^*\{m_4\} \in \underline{HH}^{\bullet,*}(\Lambda, \Lambda^\bullet)$ invertible

$$\boxed{\text{Crux } [y_\infty] \overset{!}{=} [x_\infty] \in \pi_0(X_\infty)}$$

(Muro 2020) There is an extension of the Bousfield-Kan fringed spectral sequence for computing the homotopy groups of X_∞ in terms of the tower



$$\cdots \to X_n \to \cdots \to X_1 \to X_0$$

(Bousfield-Kan 1972) There is a Milnor exact sequence of pointed sets

$$* \longrightarrow \underleftarrow{\lim}^1 \pi_1(X_i) \longrightarrow \pi_0(X_\infty) \xrightarrow{\varphi} \underleftarrow{\lim}\, \pi_0(X_i) \longrightarrow *$$

$C^{\bullet,*}(\Lambda^\bullet, m) := (HH^{\geq 2,*}(\Lambda^\bullet, \Lambda^\bullet), [m,-])$, $m = \{m_4\}$, is a cochain complex

(*) $C^{\bullet,*}(\Lambda^\bullet, m) \circlearrowleft m \cdot ?$ is a null-homotopic cochain map

invertibility of rUMP

$$m \cdot ? : HH^{\bullet,*}(\Lambda^\bullet, m) \xrightarrow{\sim} HH^{\bullet+4,-2}(\Lambda^\bullet, m) \text{ for } \bullet > 4$$

Using (*) and the extended Bousfield-Kan spectral sequence we prove

$$\underleftarrow{\lim}^1 \pi_1(X_i) = \{0\} \quad \& \quad \varphi([y_\infty]) = \varphi([x_\infty]) = * \in \underleftarrow{\lim}\, \pi_0(X_i)$$

whenever $j^*\{m_4^{y_\infty}\} = m$ ← can reduce to this case