The Donovan-Wemyss Conjecture via the Derived Auslander-Iyama Correspondence

(based on joint work with Muro and an insight of Keller)

Def (Reid 1983) $R \cong \mathbb{C}[\![x,y,z,t]\!]/(f)$

is a compound Du Val singularity if

$$f = g(x,y,z) + t\,h(x,y,z,t)$$

arbitrary

$g=0$ Kleinian / Du Val singularity

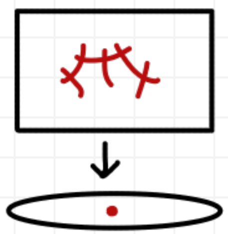
e.g. $R = \mathbb{C}[\![u,v,x,y]\!]/(uv-xy)$

simplest example

Standing assumptions

- $\operatorname{Spec} R$ has an isolated singularity
- $\exists\, p : X \longrightarrow \operatorname{Spec} R$ crepant resolution
  $p^*\omega_R \cong \omega_X$

Rmk In general, $\operatorname{Spec} R$ need not admit a crepant resolution and, if it does, it is not unique



$$X \xrightarrow{p} \operatorname{Spec} R$$

$p^{-1}(0) = \bigcup_{i=1}^{n} C_i$: exc. fibre

$C_i \cong \mathbb{P}^1$

Thm (Donovan-Wemyss 2016)

$\exists\, \Lambda(p)$: basic fin. dim. algebra that "controls" the NC deformations of $\mathcal{O}_{C_i}(-1) \in \operatorname{coh} X$

$\Lambda(p)$ is the contraction algebra of p.

e.g $R = \mathbb{C}[\![u,v,x,y]\!]/(uv-xy)$

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \operatorname{Spec} R & \end{tikzcd}$$

$\Lambda(p) \cong \Lambda(q) \cong \mathbb{C}$

misleadingly simple ...

Rmk $\Lambda(p)$ recovers all known numerical invariants associated with $p : X \to \operatorname{Spec} R$

The Donovan-Wemyss Conjecture (2016, 2020)

$p_i : X_i \to \operatorname{Spec} R_i$ crepant res. of cDV's sing $(i=1,2)$

$$R_1 \cong R_2 \iff D^b(\operatorname{mod}\Lambda(p_1)) \simeq D^b(\operatorname{mod}\Lambda(p_2))$$

($\Rightarrow$) Wemyss (2018) + Dugas (2015)

($\Leftarrow$) Reid (1983) in type $\mathbb{A}$

Recall $D_{sg}(R) := D^b(\operatorname{mod} R) / K^b(\operatorname{proj} R)$

is the singularity cat of R

- Hom-finite & Krull-Schmidt
- 2-Calabi-Yau $\operatorname{Hom}(Y, X[2]) \cong D\operatorname{Hom}(X,Y)$
- $[2] \cong \mathbb{1}$
- $\forall X \in D_{sg}(R)$ $\operatorname{End}(X)$ is symmetric since $\operatorname{Hom}(X,X) \cong \operatorname{Hom}(X,X[2]) \cong D\operatorname{Hom}(X,X)$

Def (Iyama-Yoshino 2008) $T \in D_{sg}(R)$: basic

$T \in D_{sg}(R)$ is $2\mathbb{Z}$-cluster tilting object if

$$\operatorname{add} T = \{ X \in D_{sg}(R) \mid \operatorname{Hom}(T, X[1]) = 0 \}$$

& $T \cong T[2]$

Prop (Iyama-Yoshino 2008)

$T \in D_{sg}(R)$: $2\mathbb{Z}$-cluster tilting

$$\Longrightarrow \forall X \in D_{sg}(R) \ \exists\ \underbrace{T^{-1} \to T^{0}}_{\operatorname{add} T} \to X \to T[1]$$

($\Rightarrow$ $\operatorname{thick}(T) = D_{sg}(R)$)

Thm (Wemyss 2018) There is a bijection

$$\begin{array}{ccc} \{ X \to \operatorname{Spec} R \mid \text{crepant resolution} \} /_{\cong} & & p \\ \downarrow & & \downarrow \\ \{ T \in D_{sg}(R) \mid \text{basic } 2\mathbb{Z}\text{-cluster tilting} \} /_{\cong} & & T(p) \end{array}$$

Moreover, $\operatorname{End}(T(p)) \cong \Lambda(p)$.

e.g $R = \mathbb{C}[[u,v,x,y]]/(uv - xy)$

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & \operatorname{Spec} R & \end{tikzcd}$$

$D_{sg}(R) \simeq D^b(\operatorname{mod}\mathbb{C}) / \nu[-2]$ ⟵ BMRRT Cluster cat.

$T(p) \longmapsto \mathbb{C}$

$T(q) \longmapsto \mathbb{C}[1]$

$\mathbb{C}[\nu^{\pm}]$, $|\nu| = -2$: dg algebra with trivial differential

$$\rightsquigarrow \quad D_{sg}(R) \cong D^c(\mathbb{C}[\nu^{\pm}])$$

compact / perfect der. cat.

Thm (August 2020)

$p: X \to \operatorname{Spec} R$ as above & Λ: basic fin. dim. alg.

$$D^b(\operatorname{mod} \Lambda) \simeq D^b(\operatorname{mod} \Lambda(p))$$

$$\Updownarrow$$

$\exists\, q: Y \to \operatorname{Spec} R$ crepant resolution s.t. $\Lambda \cong \Lambda(q)$

Corollary $p_i: X_i \to \operatorname{Spec} R_i$, $i = 1,2$ as in DW Conj.

$$D^b(\operatorname{mod} \Lambda(p_1)) \xleftrightarrow{\sim} D^b(\operatorname{mod} \Lambda(p_2))$$

$$\Downarrow$$

$$\exists\, q: Y \to \operatorname{Spec} R_1 \text{ s.t. } \Lambda(p_2) \cong \Lambda(q)$$

$\boxed{\text{We may assume that } \Lambda(p_1) \cong \Lambda(p_2)}$

Recall $D_{sg}(R)_{dg} := D^b(\operatorname{mod} R)_{dg} / K^b(\operatorname{proj} R)_{dg}$

is the dg singularity cat of R

$\mathcal{A}$: ess. small dg cat.

$HH^*(\mathcal{A}) := H^*(\mathbb{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}))$ is the Hochschild cohomology of $\mathcal{A}$

Thm (Hua - Keller 2018) $R \cong \mathbb{C}[[x,y,z,t]]/(f)$

$HH^0(D_{sg}(R)_{dg}) \cong T_f$: Tyurina algebra of f

Thm (Mather - Yau 1982)

T_f + dim R determine R up to isomorphism

$p_i: X_i \to \operatorname{Spec} R_i$, $i = 1,2$, as in the DW Conjecture

Suppose that $\Lambda(p_1) \cong \Lambda(p_2) =: \Lambda$

Wemyss (2018)
$$\begin{cases} T(p_1) \in D_{sg}(R_1)_{dg} \rightsquigarrow \Lambda(p_1) := \mathbb{R}\mathrm{End}(T(p_1)) \\ T(p_2) \in D_{sg}(R_2)_{dg} \rightsquigarrow \Lambda(p_2) := \mathbb{R}\mathrm{End}(T(p_2)) \end{cases}$$

$$H^*(\Lambda(p_1)) \cong H^*(\Lambda(p_2)) \cong \Lambda[z^{\pm}], \quad |z| = -2$$

(Keller 1994) $D^c(\Lambda(p_1))_{dg} \xrightarrow{\sim} D_{sg}(R_1)_{dg}$

$$D^c(\Lambda(p_2))_{dg} \xrightarrow{\sim} D_{sg}(R_2)_{dg}$$

$\boxed{D^c(\Lambda(p_1))_{dg} \simeq D^c(\Lambda(p_2))_{dg} \text{ : quasi-eq} \Longrightarrow \text{DW Conj.}}$

Thm (J-Muro 2022)

R: isolated cDV singularity & $p: X \to \operatorname{Spec} R$ crepant res.

Up to quasi-equivalence, $\mathcal{D}_{sg}(R)_{dg}$ is the unique (Karoubian) pre-triangulated dg category such that

(1) $\exists\, T \in \mathcal{D}_{sg}(R)_{dg}$ is $2\mathbb{Z}$-cluster tilting

(2) $H^*(\mathbb{R}\mathrm{End}(T)) \cong \Lambda(p)[u^{\pm}]$, $|u| = -2$

Strategy Prove $\mathbb{R}\mathrm{End}(T) \simeq \mathbb{\Lambda}(p)$ for $T \in \mathcal{A}$ satisfying (1) & (2)

Sketch of proof $\mathbb{\Lambda} = \mathbb{\Lambda}(p)$ & $\Lambda = \Lambda(p)$

$$\rightsquigarrow H^*(\mathbb{\Lambda}) \cong \Lambda[u^{\pm}], \; |u| = -2.$$

(Kadeishvili 1982)

$\mathbb{\Lambda}$ induces a minimal A_∞-algebra structure on $\Lambda[u^{\pm}]$ s.t. $\mathbb{\Lambda} \simeq \Lambda[u^{\pm}]$ as A_∞-algebras

$m_n : \Lambda[u^{\pm}]^{\otimes n} \longrightarrow \Lambda[u^{\pm}]$ of degree $2-n$, $n \geq 2$

$\rightsquigarrow \forall n \notin 2\mathbb{Z}$, $m_n = 0$

$$\boxed{\mathbb{\Lambda} = (\Lambda[u^{\pm}], m_4, m_6, m_8, \dots)}$$

$m_4 \in CC^{4,-2}(\Lambda[u^{\pm}], \Lambda[u^{\pm}])$: Hochschild cochain

$\partial_{Hoch}(m_4) = 0 \rightsquigarrow \{m_4\} \in HH^{4,-2}(\Lambda[u^{\pm}], \Lambda[u^{\pm}])$

$\{m_4\}$: Universal Massey Product (UMP)

$j : \Lambda \hookrightarrow \Lambda[u^{\pm}]$ inclusion of degree 0 component

$j^* : HH^{\bullet,\bullet}(\Lambda[u^{\pm}], \Lambda[u^{\pm}]) \longrightarrow HH^{\bullet,\bullet}(\Lambda, \Lambda[u^{\pm}])$

$j^*\{m_4\}$: restricted Universal Massey Product (rUMP)

Thm (J-Muro 2022)

$$j^*\{m_4\} \in \underline{HH}^{\bullet,\bullet}(\Lambda, \Lambda[u^{\pm}]) \text{ is a unit}$$

$\uparrow$ Hochschild-Tate cohomology

Moreover, a minimal A_∞-algebra

$$A = (\Lambda[u^{\pm}], m_4^A, m_6^A, m_8^A, \dots)$$

is A_∞-isomorphic to $\mathbb{\Lambda}$ if and only if

$$j^*\{m_4^A\} \in \underline{HH}^{\bullet}(\Lambda, \Lambda[u^{\pm}]) \text{ is a unit}$$