The Donovan–Wemyss Conjecture via the Derived Auslander–Iyama Correspondence

(joint work with Fernando Muro)

§ The Donovan–Wemyss Conjecture

(Reid 1983) $R \cong \mathbb{C}[\![x,y,z,t]\!]/(f)$ is a compound Du Val sing. if

$$f(x,y,z,t) = g(x,y,z) + t\,h(x,y,z,t)$$

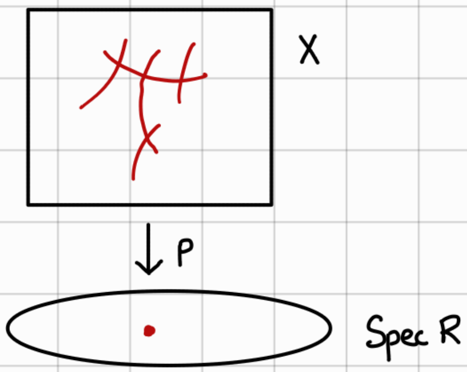
arbitrary power series

equation of a Kleinian surface singularity (e.g. $g = x^2+y^2+z^{n+1}$ in type $\mathbb{A}_n$)

Today
- R has an isolated singularity
- $\exists\ p : X \to \operatorname{Spec} R$ a crepant resolution

$\Big\}\ \circledast$

(Donovan–Wemyss 2013)



$\Lambda = \Lambda(p)$ contraction algebra of p

Idea: Exc. fibre of $p = \bigcup_{i=1}^{n} C_i$

Λ represents the functor of "simultaneous non-commutative deformations" of $C_1, \dots, C_n$ in X

Remarkable: Λ recovers all known numerical invariants of p

e.g. Toda's dimension formula for irred. contractions recovers the width (in the sense of Reid) and the Gopakumar–Vafa invariants (in the sense of Katz) in terms of $\dim_{\mathbb{C}} \Lambda$

<u>Conjecture</u> (Donovan - Wemyss 2013)

R_1, R_2 : isolated cDV's with crepant res. $p_i : X_i \to \operatorname{Spec} R_i$, $i=1,2$

$$D^b(\operatorname{mod} \Lambda(p_1)) \underset{\Delta}{\simeq} D^b(\operatorname{mod} \Lambda(p_2)) \iff R_1 \cong R_2$$

($\Leftarrow$) Follows from results of Wemyss (2018) & August (2020)

($\Rightarrow$) Known in type A (Reid 1983)

<u>Rmk</u> "DG enhanced" variants of the conjecture are known to hold
(Hua 2018, <u>Hua-Keller 2018</u>, Booth 2019)

§ Contraction algebras via $2\mathbb{Z}$-cluster tilting objects

R : isolated cDV that admits a crepant resolution

$\rightsquigarrow$ $D_{sing}(R) := D^b(\operatorname{mod} R) / K^b(\operatorname{proj} R)$ singularity category

- $\mathbb{C}$-linear triangulated category
- (R isolated) Hom-finite & Krull-Schmidt
- (dim R = 3) 2-Calabi-Yau category

$\mathbb{C}$-linear dual

$$\forall X, Y \qquad \operatorname{Hom}(X,Y)^* \cong \operatorname{Hom}(Y, X[2])$$

- (R hypersurface) 2-periodic : $[2] \cong \mathbb{1}$

Def / Thm (Iyama-Yoshino 2008, Beligiannis 2015)

$T \in D_{sing}(R)$ is $2\mathbb{Z}$-cluster tilting if the following hold:

automatic since $[2] \cong \mathbb{1}$

(1) $\mathrm{Hom}(T, T[1]) = 0$ & $T \cong T[2]$

(2) $\forall X \in D_{sing}(R)$ $\exists$ $T_1 \to T_0 \to X \to T_1[1]$ exact triangle
with $T_0, T_1 \in \mathrm{add}\, T$

idempotent complete additive closure of T

Rmk $T \in D_{sing}(R)$: $2\mathbb{Z}$-cluster tilting $\Rightarrow$ $\mathrm{thick}(T) = D_{sing}(R)$

Thm (Wemyss 2018) There is a bijective correspondence between:

(1) Crepant resolutions of R $/ \cong$

(2) Basic $2\mathbb{Z}$-cluster tilting objects in $D_{sing}(R)$ $/ \cong$

Moreover, if $T = T(p)$ for a crepant res. $p: X \to \mathrm{Spec}\, R$, then

$\Lambda(p) \cong \mathrm{End}(T)$ ← ordinary endomorphism alg. of T

Thm (August 2020) $p: X \to \mathrm{Spec}\, R$ crepant resolution

Λ': basic fin. dim. algebra. TFAE

(1) $D^b(\mathrm{mod}\, \Lambda') \xrightarrow[\Delta]{\sim} D^b(\mathrm{mod}\, \Lambda(p))$

(2) $\exists\, T \in D_{sing}(R)$: $2\mathbb{Z}$-cluster tilting such that $\Lambda' \cong \mathrm{End}(T)$

§ The DG singularity category determines R

$R = \mathbb{C}[[x,y,z,t]]/(f)$: isolated cDV that admits a crepant resolution

$\rightsquigarrow$ $D_{sing}(R)_{dg} := D^b(\mathrm{mod}\,R)_{dg} / K^b(\mathrm{proj}\,R)_{dg}$ DG singularity category

Drinfeld quotient

Thm (Hua-Keller 2018) There is an isomorphism of algebras

$$HH^0(D_{sing}(R)_{dg}) \cong \underbrace{\mathbb{C}[[x,y,z,t]]/(f, \partial_x f, \partial_y f, \partial_z f, \partial_t f)}_{\text{Tyurina algebra of } f}$$

($\xRightarrow[1982]{\text{Mather-Yau}}$ determines R up to isomorphism since dim R = 3 is fixed)

Pseudo-proof of the DW conjecture (after Keller)

R_1, R_2 : isolated cDV's with crepant res. $p_i : X_i \to \mathrm{Spec}\, R_i$, $i=1,2$

Suppose that $D^b(\mathrm{mod}\,\Lambda(p_1)) \underset{\Delta}{\simeq} D^b(\mathrm{mod}\,\Lambda(p_2))$

(Wemyss 2018) $\exists$ $\left.\begin{array}{l} T_1 \in D_{sing}(R_1) \\ T_2' \in D_{sing}(R_2) \end{array}\right\}$ $2\mathbb{Z}$-cluster tilting

with $\Lambda(p_1) \cong \mathrm{End}(T_1)$ & $\Lambda(p_2) \cong \mathrm{End}(T_2')$

(August 2020) $\exists$ $T_2 \in D_{sing}(R_2)$: $2\mathbb{Z}$-cluster tilting
such that $\mathrm{End}(T_2) \cong \Lambda(p_1)$

Set $\Lambda := \Lambda(p_1) \cong \operatorname{End}_{D_{sing}(R_1)}(T_1) \cong \operatorname{End}_{D_{sing}(R_2)}(T_2)$

We introduce the derived contraction algebra

$$\boldsymbol{\Lambda}_1 := \mathbb{R}\mathrm{End}(T_1) \quad \& \quad \boldsymbol{\Lambda}_2 := \mathbb{R}\mathrm{End}(T_2)$$

Keller 1994 $\rightsquigarrow$

$$\begin{tikzcd}
T_1 \arrow[r, mapsto] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \boldsymbol{\Lambda}_1 \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \\
D_{sing}(R_1)_{dg} \arrow[r, phantom, "\simeq"] & D^c(\boldsymbol{\Lambda}_1)_{dg} \arrow[d, phantom, "\cong ?"] \arrow[r, rightsquigarrow] & HH^0(\boldsymbol{\Lambda}_1) \cong \text{Tyurina of } R_1 \arrow[d, phantom, "\cong ?"] \\
D_{sing}(R_2)_{dg} \arrow[r, phantom, "\simeq"] & D^c(\boldsymbol{\Lambda}_2)_{dg} & HH^0(\boldsymbol{\Lambda}_2) \cong \text{Tyurina of } R_2 \\
T_2 \arrow[r, mapsto] \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & \boldsymbol{\Lambda}_2 \arrow[u, phantom, "\rotatebox{90}{$\in$}"] &
\end{tikzcd}$$

Notice $H^*(\boldsymbol{\Lambda}_1) \cong H^*(\boldsymbol{\Lambda}_2) \cong \underbrace{\Lambda[z^{\pm}]}_{\text{Laurent polynomials}}$ where $|z| = -2$

The conjecture follows once we know that $\boldsymbol{\Lambda}_1 \underset{q\text{-iso}}{\simeq} \boldsymbol{\Lambda}_2$ $\blacksquare$

§ Contraction algebras are determined by their cohomology

R : isolated cDV with crepant resolution $p : X \longrightarrow \operatorname{Spec} R$

$\rightsquigarrow$ $\Lambda := \Lambda(p)$: contraction algebra

immediately implies the conjecture

Thm (J - Muro 2022)

Up to quasi-isomorphism, there exists a unique DGA $\boldsymbol{\Lambda}$ such that:

(1) $H^*(\boldsymbol{\Lambda}) \cong \Lambda[z^{\pm}]$, $|z| = -2$

(2) $\boldsymbol{\Lambda} \in D^c(\boldsymbol{\Lambda})$ is a $2\mathbb{Z}$-cluster tilting object

Let $\Lambda = \mathbb{R}\mathrm{End}(T(p))$ the derived contraction algebra of p

$$\Lambda^* := \Lambda[u^\pm], \quad |u| = -2 \qquad \text{so that} \qquad H^*(\Lambda) \cong \Lambda^*$$

Kadeishvili 1982 $\rightsquigarrow$ $(\Lambda^*, m_4, m_6, \dots, m_{2k}, \dots)$ minimal A_∞-structure

such that $\Lambda \underset{qiso}{\simeq} \Lambda^*$ as A_∞-algebras

Recall $m_p : (\Lambda^*)^{\otimes p} \longrightarrow \Lambda^*$ of degree $2-p$

as a graded algebra!

$\rightsquigarrow$ $m_4 \in C^{4,-2}(\Lambda^*, \Lambda^*)$: Hochschild complex

$$\partial_{Hoch}(m_4) = 0 \quad \rightsquigarrow \quad \{m_4\} \in HH^{4,-2}(\Lambda^*, \Lambda^*)$$

Universal Massey Product (of length 4)

$$j : \Lambda \xrightarrow{\deg 0} \Lambda^* \quad \rightsquigarrow \quad j^* : HH^{4,-2}(\Lambda^*, \Lambda^*) \longrightarrow HH^{4,-2}(\Lambda, \Lambda^*)$$
$$\{m_4\} \longmapsto j^*\{m_4\}$$

restricted UMP

degree -2 part

$$j^*\{m_4\} \in HH^{4,-2}(\Lambda, \Lambda^*) \cong \mathrm{Ext}^4_{\Lambda^e}(\Lambda, \Lambda u)$$

$$\rightsquigarrow j^*\{m_4\} = [0 \to \Lambda \to X \to \underbrace{P_2 \to P_1 \to P_0}_{\text{projective-injective } \Lambda\text{-bimodules}} \to \Lambda \to 0]$$

Prop (Muro 2022) TFAE for a class $\alpha \in \mathrm{Ext}^4_{\Lambda^e}(\Lambda, \Lambda)$

(1) $\alpha = [0 \to \Lambda \to \underbrace{P_3 \to P_2 \to P_1 \to P_0}_{\text{projective-injective } \Lambda\text{-bimodules}} \to \Lambda]$

(2) α is a unit in $\underline{HH}^{\bullet,*}(\Lambda, \Lambda^*)$ (Hochschild-Tate cohomology)

Thm (J-Muro 2022) $\mathbb{B}$: DGA with $H^*(\mathbb{B}) \cong \Lambda^* \cong H^*(\mathbb{\Lambda})$. TFAE

(1) $j^*\{m_4^{\mathbb{B}}\} \in \underline{HH}^{\bullet,*}(\Lambda, \Lambda^*)$ is a unit.

(2) $\mathbb{B} \in D^c(\mathbb{B})$ is $2\mathbb{Z}$-cluster tilting.

Coro $\mathbb{\Lambda}$: derived contraction algebra $\Rightarrow$ $j^*\{m_4\} \in HH^{\bullet,*}(\Lambda, \Lambda^*)$ unit

Thm (J-Muro 2022)

Up to quasi-isomorphism, there exists a unique DGA $\mathbb{\Lambda}$ such that:

(1) $H^*(\mathbb{\Lambda}) \cong \Lambda[z^{\pm}]$, $|z| = -2$

(2) $j^*\{m_4\} \in \underline{HH}^{\bullet,*}(\Lambda, \Lambda^*)$ is a unit

Coro TFAE

(1) $\mathbb{\Lambda}$ is formal (i.e $\mathbb{\Lambda} \cong \Lambda^*$ ← trivial DG/A_∞-algebra structure)

(2) $\Lambda = H^0(\mathbb{\Lambda}) \cong \mathbb{C}$

(3) $R = \mathbb{C}[[x,y,z,t]]/(xy - zt)$ is the base of the Atiyah flop.