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The Triangulated Auslander–Iyama Correspondence

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Reconstruction theorems

Setup and motivation

$\mathbf{k}$: perfect field

$\mathcal{T}$: $\mathbf{k}$-linear triangulated cat

- idempotent complete
- Hom-finite

$G \in \mathcal{T}$: (basic) classical generator

Aim: Reconstruct $\mathcal{T}$ from G as a triangulated category

In general,

$$\mathcal{T} \not\simeq \mathrm{D}^c(\mathcal{T}(G,G))$$

and

$$\mathcal{T} \not\simeq \mathrm{D}^c(\mathcal{T}(G,G)^\bullet)$$

where

$$\mathcal{T}(G,G)^\bullet = \bigoplus_{i \in \mathbb{Z}} \mathcal{T}(G, G[i])$$

Tautological solution via DG algebras

$\mathcal{T}$: algebraic triangulated category

Keller (1994) $\exists\, A$ DG algebra s.t.

$$\begin{aligned} \mathcal{T} &\xrightarrow{\simeq} \mathrm{D}^{\mathrm{c}}(A) \\ G &\longmapsto A \end{aligned}$$

and $H^{\bullet}(A) \cong \mathcal{T}(G,G)^{\bullet}$

Q: When is A unique up to qiso?

Meta Theorem

A, B: DG algs with $H^{\bullet}(A) \cong H^{\bullet}(B)$

$$(?) \implies A \simeq B \implies \mathrm{D}^{\mathrm{c}}(A) \simeq \mathrm{D}^{\mathrm{c}}(B)$$

Condition (?) should be

- of elementary character
- sufficiently general
- valid in interesting examples

$d\mathbb{Z}$-cluster tilting objects ($d \geq 1$)

Suppose that A satisfies

1. $\varphi\colon A \cong A[d]$ in $\mathsf{D}^c(A)$
2. $i \notin d\mathbb{Z} \implies H^i(A) = 0$

Then

$$H^\bullet(A) \cong \bigoplus_{di \in d\mathbb{Z}} {}_{\sigma^i}\Lambda_1$$

where

$$\Lambda := H^0(A) \quad \text{and} \quad \sigma(a) = \varphi^{-1} a \varphi$$

Iyama–Yoshino (2008) Suppose now that $H^0(A)$ is finite-dimensional and

3. $M \in \operatorname{add}(A) \Leftrightarrow \operatorname{Hom}_A(A, M[k]) = 0$
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for $0 < k < d$.

We say that $A \in \mathsf{D}^c(A)$ is a $d\mathbb{Z}$-cluster tilting object if it satisfies 1–3.

Rmk: $1\mathbb{Z}$-CT $\iff$ $\operatorname{add}(A) = \mathsf{D}^c(A)$

… more on $d\mathbb{Z}$-cluster tilting objects

$A \in \mathsf{D}^{\mathsf{c}}(A)$ $d\mathbb{Z}$-cluster tilting

$\Lambda = H^0(A)$ basic finite-dimensional

Heller (1966), Freyd (1968), Geiß–Keller–Oppermann (2013)

Λ is a Frobenius algebra

Green–Snashall–Solberg (2003), Hanihara (2020)

Λ is twisted $(d+2)$-periodic w.r.t. σ,

$$\Omega^{d+2}_{\Lambda^e}(\Lambda) \simeq {}_1\Lambda_\sigma \quad \text{in} \quad \underline{\mathrm{mod}}(\Lambda^e)$$

The Triangulated Auslander–Iyama Correspondence

There are bijective correspondences between the following objects:

DG algebras A such that $A \in \mathsf{D}^c(A)$ is $d\mathbb{Z}$-cluster tilting
up to quasi-isomorphism

Pairs $(\mathcal{T}, G)$ where $G \in \mathcal{T}$ is $d\mathbb{Z}$-cluster tilting and $\mathcal{T}$ is algebraic
up to exact equivalence preserving $\mathsf{add}(()G)$

Pairs (Λ, σ) where Λ is twisted $(d+2)$-periodic w.r.t σ
up to algebra isomorphism preserving $[\sigma] \in \mathsf{Out}(\Lambda)$

Rmk: The case $d = 1$, where $\mathsf{add}(A) = \mathsf{D}^c(A)$, is due to Muro

The Donovan–Wemyss Conjecture

Compound Du Val singularities

Reid (1983) $R \cong \mathbb{C}[\![u,v,x,t]\!]/(f)$ is a compound Du Val singularity (cDV) if

$$f(u,v,x,t) = g(u,v,x) + t \cdot h(u,v,x,t)$$

and $g = 0$ is the equation of a Kleinian surface singularity, e.g.

$$g(u,v,x) = u^2 + v^2 + x^{n+1} \qquad (\text{type } \mathbb{A}_n)$$

- Fundamental class of 3-dimensional singularities
- Important in the Minimal Model Program for 3-folds

Today: Focus on case where

- R has isolated singularities
- $\operatorname{Spec}(R)$ admits a resolution

Contraction algebras

$$p\colon Y \to X = \operatorname{Spec}(R) \quad \text{crepant resolution} \qquad F = p^{-1}(o) \quad \text{red exc fibre}$$

$$\text{crepant: } p^*(\omega_X) = \omega_Y$$

$$\text{resolution: } Y \text{ is smooth and } p \text{ is birational, isomorphism in } Y \setminus F$$

$$\implies \qquad F = C_1 \cup \cdots \cup C_n \text{ is a tree of rational curves in } Y$$

Donovan–Wemyss (2013) The non-commutative deformations of

$$\mathcal{O}_{C_1} \oplus \cdots \oplus \mathcal{O}_{C_1} \in \operatorname{coh}(Y)$$

are represented by a finite-dimensional algebra $\Lambda = \Lambda(p)$, the contraction algebra

The Classification Conjecture

$$p_1 \colon Y_1 \to X_1 = \mathrm{Spec}(R_1) \quad \text{and} \quad p_2 \colon Y_2 \to X_2 = \mathrm{Spec}(R_2)$$

crepant resolutions of isolated cDV singularities

$\Lambda_1 = \Lambda(p_1)$ and $\Lambda_2 = \Lambda(p_2)$ their contraction algebras

Donovan–Wemyss (2013) $\quad \mathrm{D}^c(\Lambda_1) \simeq \mathrm{D}^c(\Lambda_2) \quad \overset{?}{\Longrightarrow} \quad R_1 \cong R_2.$

Wemyss (2018) + Dugas (2015) Different p's yield derived eq contraction algebras

Contraction algebras via $2\mathbb{Z}$-cluster tilting objects

R isolated cDV singularity with crepant resolution

$\mathrm{D}_{\mathrm{sg}}(R) = \mathrm{D}^{\mathrm{b}}(\operatorname{mod} R)/\mathrm{D}^{\mathrm{c}}(R)$ singularity category

$\Longrightarrow$ $\mathrm{D}_{\mathrm{sg}}(R)$ is an algebraic $\mathbb{C}$-linear triangulated cat with split idempotents

Moreover, $\mathrm{D}_{\mathrm{sg}}(R)$ is 2-periodic: $[2] \cong \mathbf{id}$

Wemyss (2018) The endomorphism algebras of $2\mathbb{Z}$-cluster tilting objects in $\mathrm{D}_{\mathrm{sg}}(R)$ are precisely the contraction algebras of R

August (2020) The contraction algebras of R form a single and complete derived equivalence class of algebras

Proof of the Classification Conjecture

R isolated cDV singularity with crepant resolution

$\mathcal{D}_{\mathrm{sg}}(R)$ canonical DG enhancement of $\mathrm{D}_{\mathrm{sg}}(R)$

Hua–Keller (2018) The DG category $\mathcal{D}_{\mathrm{sg}}(R)$ determines R up to isomorphism:

$$HH^0(\mathcal{D}_{\mathrm{sg}}(R)) \cong \frac{\mathbb{C}[\![u,v,x,t]\!]}{\left(f, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t}\right)}$$

is the Tyurina algebra of R, which determines R since we know $\dim R$

Triangulated Auslander–Iyama Correspondence $\mathrm{D}_{\mathrm{sg}}(R)$ is uniquely determined as an enhanced triangulated category by any contraction algebra of R

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Extra: The key theorem

$$\Lambda \quad \text{twisted } (d+2)\text{-periodic w.r.t. } \sigma$$

$$\Lambda(\sigma,d) = \bigoplus_{di\in d\mathbb{Z}} {}_{\sigma^i}\Lambda_1, \qquad a\cdot b = \sigma^j(a)b, \quad |b| = dj$$

$$j^*\colon HH^{\bullet,d\star}(\Lambda(\sigma,d),\Lambda(\sigma,d)) \longrightarrow HH^{\bullet,d\star}(\Lambda,\Lambda(\sigma,d)) \cong \operatorname{Ext}^{\bullet}_{\Lambda^e}(\Lambda,{}_{\sigma^\star}\Lambda_1)$$

J–Muro There exists an essentially unique minimal A_∞-algebra structure

$$(\Lambda(\sigma,d),\ m_{d+2},\ m_{2d+2},\ m_{3d+2},\ \ldots)$$

such that $j^*\{m_{d+2}\} \in HH^{d+2,-d}(\Lambda,\Lambda(\sigma,d))$ is a unit in $\underline{HH}^{\bullet,\star}(\Lambda,\Lambda(\sigma,d))$