LUND UNIVERSITY

Brackets, trees and the Borromean rings

Gustavo Jasso

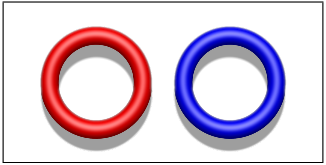
Centre for Mathematical Sciences

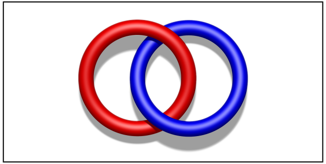
A bird's-eye view on algebraic structures

Mathematical objects	- - - →	Algebraic structures
X – topological space		$\pi_1(X, x)$ – fundamental **group**
M – smooth manifold		$\Omega(M)$ – **algebra** of differential forms
V – complex variety		$\mathbb{C}[V]$ – coordinate **ring**
G – Lie group		$\mathrm{Lie}(G)$ – **Lie algebra**

Properties of the mathematical objects
should be reflected in properties of the associated algebraic structures

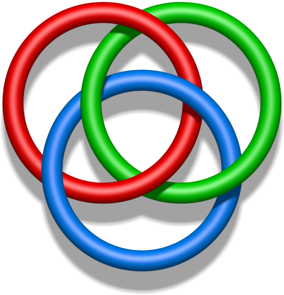
Q: What differentiates these configurations from each other?





A: The circles in the Hopf link (bottom) are, well, linked.

Consider now the Borromean rings:



Q: What differentiates them from three unlinked circles?

Brackets and trees

The combinatorics of the associativity equation

The familiar associativity equation

$$(ab)c = a(bc)$$

is the source of rich combinatorics:

$$\begin{aligned} 1 &: (ab) \\ 2 &: (ab)c,\ a(bc) \\ 5 &: ((ab)c)d),\ ((a(bc))d),\ a((bc)d), \\ &\quad a(b(cd)),\ (ab)(cd) \\ 14 &: a(b(c(de)),\ a(b((cd),e),\ \dots \end{aligned}$$

Theorem (Catalan 1838)

There are precisely

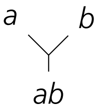
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

different ways to correctly parenthesise a word on $n+1$ letters.

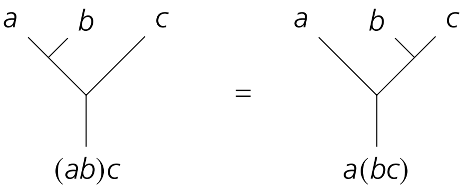
The number C_n is called the n-th Catalan number: 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, …

From associativity to dendrology

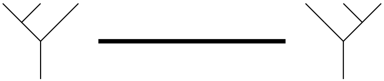
Depict the binary operation as a tree:



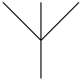
The associativity equation becomes



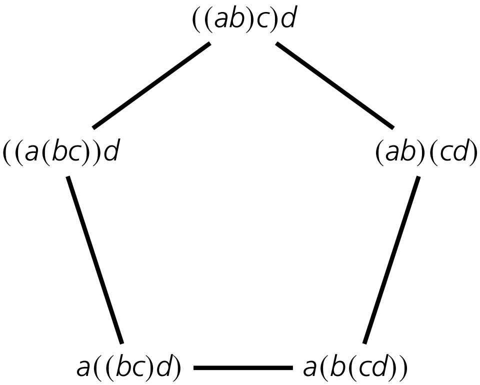
The previous equation suggests a *local* transformation on a binary tree, called a flip:

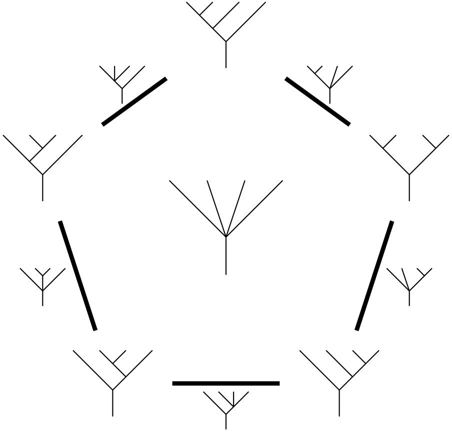


Observe that contracting the unique internal edge in the above trees yields the (non-binary!) tree



From associativity to dendrology II





The Associahedron

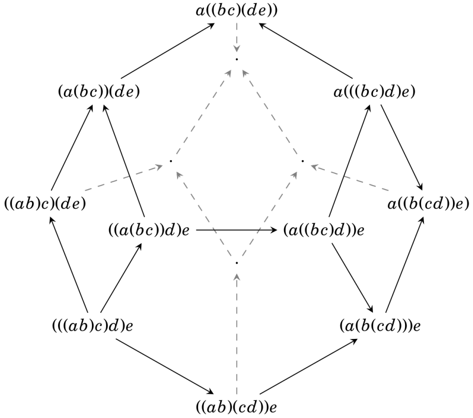
Theorem (Tamari 1951, Stasheff 1963, Loday 2004)

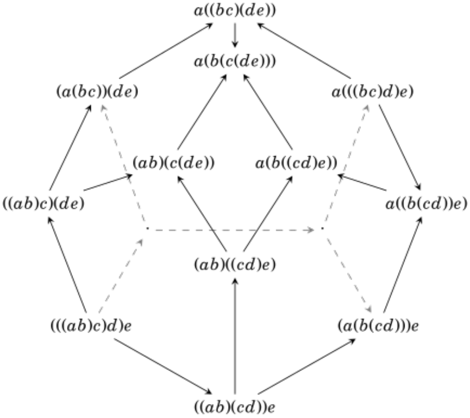
There is a convex polytope K_{n+1} of dimension $n-1$ whose k-dimensional cells are in bijection with planar rooted trees with $n+1$ leaves and $n-k$ internal vertices.

Moreover, there is an edge between two vertices if and only if the corresponding binary trees are related by a single flip.

K_{n+1} is called the Associahedron of dimension $n-1$

The Associahedron K_5

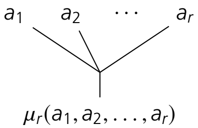




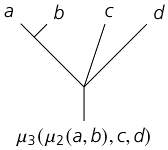
Exercise: Label the faces of K_5 with planar trees.

Planar trees as multi-operations

Interpret the rooted corolla with r leaves as an operation with r inputs:



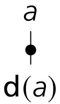
By combining these, each planar rooted tree yields a multi-operation:



We also consider a unary operation

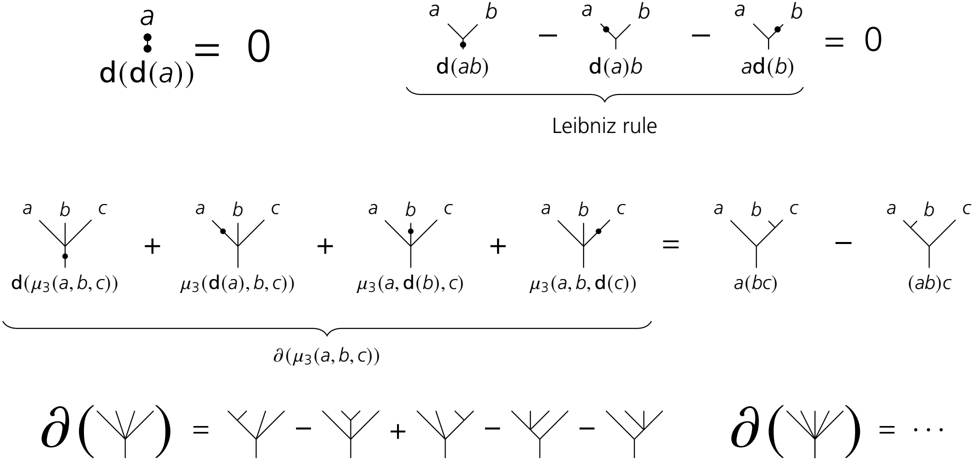
$$a \longmapsto \mu_1(a) =: \mathbf{d}(a)$$

corresponding to a planar rooted tree with one leaf:



We decorate the tree with a $\bullet$ to remember that something happens as we flow through the tree

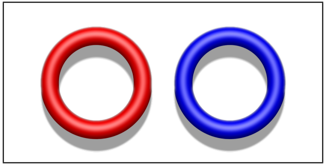
The A_∞-equations (Stasheff 1963)

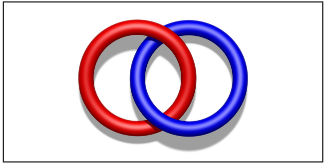


In general, the RHS is indexed by the boundary of the associahedron.

Back to the Borromean rings

Q: What differentiates these configurations from each other?





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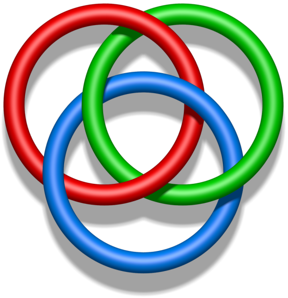


Q: What differentiates them from three unlinked circles?

The Hopf link, revisited

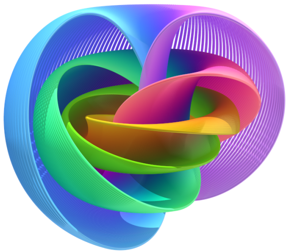
Let X be the complement of the Hopf link in the 3-sphere. The singular cohomology of X is

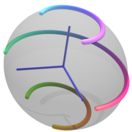
$$H^0(X;\mathbb{R}) \cong \mathbb{R},$$
$$H^1(X;\mathbb{R}) \cong \mathbb{R}\alpha \oplus \mathbb{R}\beta,$$
$$H^2(X;\mathbb{R}) \cong \mathbb{R}.$$

Moreover, the cup product $\mu_2(\alpha,\beta) \neq 0$.

The complement of two unlinked circles has isomorphic cohomology, but

$$\mu_2(\alpha,\beta) = 0.$$





A_∞-structure on singular cohomology

The singular cohomology

$$H^*(X;\mathbb{R}) = \bigoplus_{p=0}^{\infty} H^p(X;\mathbb{R})$$

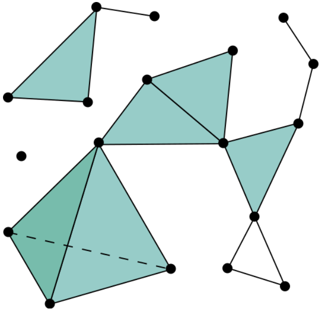
of a space X is endowed with multi-operations

$$\mathbf{d} = \mu_1, \mu_2, \mu_3, \ldots$$

that satisfy the A_∞-equations, where

$$\mathbf{d} = 0 \quad \text{and} \quad \mu_2 = \text{cup product.}$$

What do the higher operations tell us about the space X?



Massey's beautiful theorem

Let X be the complement of the Borromean rings in the 3-sphere.

$$H^0(X;\mathbb{R}) \cong \mathbb{R},$$
$$H^1(X;\mathbb{R}) \cong \mathbb{R}\alpha \oplus \mathbb{R}\beta \oplus \mathbb{R}\gamma,$$
$$H^2(X;\mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R}.$$

However,

$$\mu_2(\alpha,\beta) = \mu_2(\alpha,\gamma) = \mu_2(\beta,\gamma) = 0$$

and the same is true for the complement of three unlinked circles.

Theorem (Massey 1969)

Let X be the complement of the Borromean rings in the 3-sphere. Then,

$$\mu_3(\alpha,\beta,\gamma) \neq 0.$$

In the case of the complement of three unlinked circles one has

$$\mu_3(\alpha,\beta,\gamma) = 0.$$

First (?) proof that the Borromean rings are non-trivial.

A_∞-structures in my research

The Donovan–Wemyss Conjecture

Compound Du Val singularities + cond. $\dashrightarrow$ (Donovan–Wemyss) Finite-dim algebras

X – cDV singularity

$\Lambda(X,p)$ – **contraction algebra**

(Donovan–Wemyss 2013) $\quad \Lambda(X,p) \overset{\text{der}}{\sim} \Lambda(Y,q) \overset{?}{\Longrightarrow} X \cong Y$

(Wemyss 2018, Hua–Keller 2018, August 2020)
Reduced the DW Conjecture to an algebraic problem

2022: Algebraic problem solved with **Fernando Muro** using A_∞-structures!

The conjecture holds!

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