The Triangulated Auslander–Iyama Correspondence

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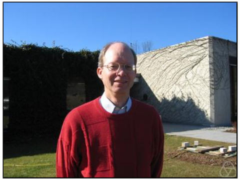
The periodicity conjecture for pairs of Dynkin diagrams

By Bernhard Keller

Abstract

We prove the periodicity conjecture for pairs of Dynkin diagrams using Fomin-Zelevinsky's cluster algebras and their (additive) categorification via triangulated categories.



Bernhard in 2008

Myself at ICONCART

Morita Theory

Classical Morita Theory (1958)

Let A and B be algebras. The following statements are equivalent:

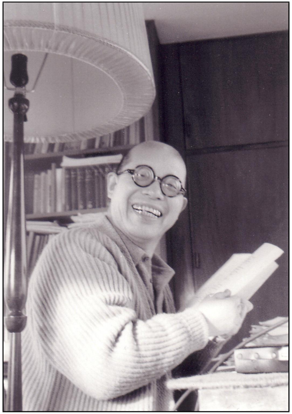
- There exists an equivalence

$$\operatorname{Mod} A \xrightarrow{\simeq} \operatorname{Mod} B.$$

- There exists $P \in \operatorname{proj} A$ a generator such that $\operatorname{End}_A(P) \cong B$.

Gabriel (1962), Freyd (1966)

The second property characterises module categories among **cocomplete abelian categories**.

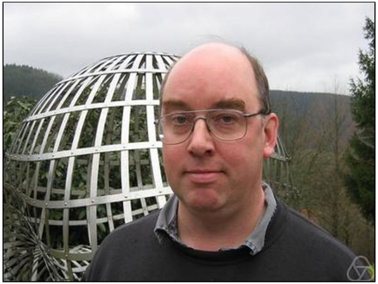


Kiiti Morita

Rickard's Derived Morita Theory (1989)

Let A and B be algebras. The following statements are equivalent:

- There exists an exact equivalence
$$D(\operatorname{Mod} A) \xrightarrow{\simeq} D(\operatorname{Mod} B).$$
- There exists $P \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ a generator with
$$\bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_A(P, P[i]) = \operatorname{Hom}_A(P, P) \cong B.$$



Jeremy Rickard in 2006

Q: How to characterise derived module categories?

Keller's Differential Graded Morita Theory (1994)

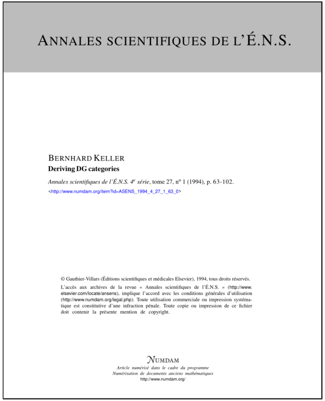
Let A and B be **DG** algebras. The following statements are equivalent:

- There exists a quasi-equivalence

$$D(A)_{\mathrm{dg}} \xrightarrow{\simeq} D(B)_{\mathrm{dg}}.$$

- There exists $P \in \mathrm{D}^{\mathrm{c}}(A)$ a generator such that $\mathbf{REnd}_A(P) \simeq B$.

The second property characterises derived categories of DG algebras among **cocomplete algebraic triangulated categories**.



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Deriving DG categories

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Correspondences of Morita–Tachikawa Type

The Dominant Dimension

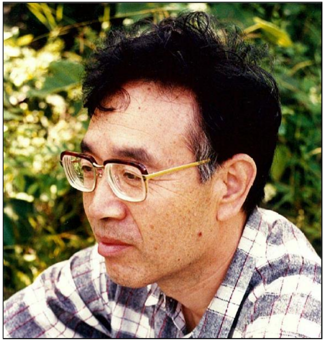
Tachikawa (1964) The **dominant dimension** of a finite-dimensional algebra Γ is

$$\operatorname{domdim} \Gamma = \sup\{\, i \geq 0 \mid Q^0, Q^1, \dots, Q^{i-1} \in \operatorname{proj} \Gamma \,\}$$

where

$$0 \to \Gamma_\Gamma \to Q^0 \to Q^1 \to \cdots$$

is a minimal injective coresolution.



Hiroyuki Tachikawa

Nakayama Conjecture (1958) $\operatorname{domdim} \Gamma = \infty \implies \Gamma$ is self-injective

The Morita–Tachikawa Correspondence (1970, 1971)

The map $(A, M_A) \mapsto \mathsf{End}_A(M)$ induces a bijective correspondence between:

1. Pairs (A, M_A) where A is a finite-dimensional algebra and $A \oplus DA \in \mathsf{add}(M)$ up to Morita equivalence (of pairs).
2. Finite-dimensional algebras Γ such that $\operatorname{domdim} \Gamma \geq 2$ up to Morita equivalence.

Paradigm: Relate further properties of $M \in \operatorname{mod} A$ to properties of Γ and vice versa.

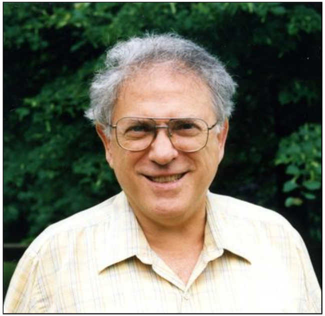
The Auslander Correspondence (1971)

The map $(A, M_A) \mapsto \mathrm{End}_A(M)$ induces a bijective correspondence between:

1. Pairs (A, M_A) where A is a finite-dimensional algebra and $\mathrm{add}(M) = \mathrm{mod}\, A$ up to Morita equivalence (of pairs).
2. Finite-dimensional algebras Γ such that

$$\mathrm{domdim}\, \Gamma \geq 2 \geq \mathrm{gldim}\, \Gamma$$

   up to Morita equivalence.



Maurice Auslander in 1987

Rmk: This result led Auslander and Reiten to develop the theory of almost split sequences throughout the 1970s.

Cluster Tilting Modules ($d \geq 1$)

Iyama (2007) $M \in \operatorname{mod} A$ is d**-cluster tilting** if

$$\operatorname{add}(M) = \{ L \in \operatorname{mod} A \mid \forall 0 < k < d \ \operatorname{Ext}_A^k(L, M) = 0 \}$$
$$\operatorname{add}(M) = \{ N \in \operatorname{mod} A \mid \forall 0 < k < d \ \operatorname{Ext}_A^k(M, N) = 0 \}$$

Osamu Iyama in 2014

Rmk: $M \in \operatorname{mod} A$ is 1-cluster tilting $\iff$ $\operatorname{add}(M) = \operatorname{mod} A$

Iyama–Yoshino (2008) Same definition works well in triangulated categories and

d-cluster tilting objects are **(classical) generators**

The Auslander–Iyama Correspondence (2007)

The map $(A, M_A) \mapsto \operatorname{End}_A(M)$ induces a bijective correspondence between:

1. Pairs (A, M_A) where A is a finite dimensional algebra and M is a d-cluster tilting module up to Morita equivalence (of pairs).
2. Finite-dimensional algebras Γ such that

$$\operatorname{domdim} \Gamma \geq d + 1 \geq \operatorname{gldim} \Gamma$$

up to Morita equivalence.

Rmk: This is one of the seminal results in Iyama's higher Auslander–Reiten Theory.

Derived Correspondences

A Motivating Question

$\mathcal{T}$ – cocomplete **algebraic** triangulated category

$G \in \mathcal{T}^c$ – compact generator

Keller (1994) There exists a DG algebra A and an exact equivalence

$$\mathcal{T} \xrightarrow{\simeq} D(A), \qquad G \longmapsto A; \qquad \bigoplus_{i\in\mathbb{Z}} \mathcal{T}(G, G[i]) \cong H^\bullet(A).$$

Q: When is A determined up to quasi-isomorphism by

$H^0(A) = \mathrm{End}_A(A)$ + minimal additional data?

Twisted Periodic Algebras

From now on we work over a **perfect** field!

A finite-dimensional algebra Λ is **twisted** ***n*****-periodic** w.r.t. $\sigma \in \mathrm{Aut}(\Lambda)$ if there exists an exact sequence of Λ-bimodules

$$0 \to {}_1\Lambda_\sigma \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \Lambda \to 0$$

with $P_0, P_1, \ldots, P_{n-1} \in \operatorname{proj}\Lambda$. $\quad (\iff \Omega^n_{\Lambda^e}(\Lambda) \simeq {}_1\Lambda_\sigma \quad)$

Green–Snashall–Solberg (2013) Self-injective algebras of finite representation type are twisted periodic.

The Triangulated Auslander–Iyama Correspondence

We say that $A \in \mathrm{D}^c(A)$ is **$d\mathbb{Z}$-cluster tilting** if it is d-cluster tilting and $A \cong A[d]$.

The map $A \mapsto (H^0(A), \varphi\colon A \cong A[d])$ induces a bijective correspondence between:

1. DG algebras A such that $H^0(A)$ is basic finite-dimensional and $A \in \mathrm{D}^c(A)$ is $d\mathbb{Z}$-cluster tilting, up to quasi-isomorphism.
2. Pairs (Λ, σ) such that Λ is basic twisted $(d+2)$-periodic w.r.t σ, up to algebra isomorphisms preserving $[\sigma] \in \mathrm{Out}(\Lambda)$.

Rmk: The case $d = 1$, when $\mathrm{add}(A) = \mathrm{D}^c(A)$, is due to Muro.

From $d\mathbb{Z}$-cluster tilting to Hochschild cohomology

$$A \in \mathrm{D}^{\mathrm{c}}(A)\ d\mathbb{Z}\text{-cluster tilting} \quad \& \quad \varphi\colon A \xrightarrow{\simeq} A[d]$$

$$\Lambda := H^0(A), \qquad \sigma\colon a \longmapsto \varphi^{-1}a\varphi$$

$$H^\bullet(A) \cong \Lambda(\sigma, d) := \bigoplus_{di \in d\mathbb{Z}} {}_{\sigma^i}\Lambda_1, \qquad a * b := \sigma^j(a)b, \quad |b| = dj,$$

Geiss–Keller–Oppermann (2013) + GSS (2003) + Hanihara (2020)

$$\exists\, \eta\colon \quad 0 \to {}_1\Lambda_\sigma \to P_{d+1} \to P_d \to \cdots \to P_1 \to P_0 \to \Lambda \to 0$$

$$[\eta] \in \mathrm{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma) = \mathrm{HH}^{d+2,-d}(\Lambda, \Lambda(\sigma, d))$$

The Key Theorem

$$\Lambda \quad \text{twisted } (d+2)\text{-periodic w.r.t. } \sigma$$

$$\Lambda(\sigma, d) = \bigoplus_{di \in d\mathbb{Z}} {}_{\sigma^i}\Lambda_1, \qquad a * b = \sigma^j(a)b, \quad |b| = dj$$

$$j^* \colon \mathrm{HH}^{\bullet, d\star}(\Lambda(\sigma,d), \Lambda(\sigma,d)) \longrightarrow \mathrm{HH}^{\bullet, d\star}(\Lambda, \Lambda(\sigma,d)) \cong \mathrm{Ext}^{\bullet}_{\Lambda^e}(\Lambda, {}_{\sigma^\star}\Lambda_1)$$

J–Muro (2022) There exists an essentially unique minimal A_∞-algebra structure

$$(\Lambda(\sigma,d),\ m_{d+2},\ m_{2d+2},\ m_{3d+2},\ \dots)$$

such that $j^*\{m_{d+2}\} \in \mathrm{HH}^{d+2,-d}(\Lambda, \Lambda(\sigma,d))$ is a unit in $\underline{\mathrm{HH}}^{\bullet,\star}(\Lambda, \Lambda(\sigma,d))$

The Donovan–Wemyss Conjecture

Compound Du Val Singularities

Reid (1983) $R \cong \mathbb{C}[\![u,v,x,t]\!]/(f)$ is a compound Du Val singularity (cDV) if

$$f(u,v,x,t) = g(u,v,x) + t \cdot h(u,v,x,t)$$

and $g = 0$ is the equation of a Kleinian surface singularity, e.g.

$$g(u,v,x) = u^2 + v^2 + x^{n+1} \qquad (\text{type } \mathbb{A}_n)$$

R isolated cDV singularity with crepant resolution

$\mathrm{D}_{\mathrm{sg}}(R) = \mathrm{D}^{\mathrm{b}}(\operatorname{mod} R)/\mathrm{K}^{\mathrm{b}}(\operatorname{proj} R)$ singularity category

$\mathrm{D}_{\mathrm{sg}}(R)$ is 2-periodic: $[2] \cong \mathbf{id}$

The Donovan–Wemyss Conjecture

Wemyss (2018) The endomorphism algebras of $2\mathbb{Z}$-cluster tilting objects in $\mathrm{D}_{\mathrm{sg}}(R)$ are (precisely) the **contraction algebras** of R

Donovan–Wemyss (2013) Defined contraction algebras using non-commutative deformation theory.

R_1 and R_2 isolated cDV singularities with crepant resolutions

Λ_1 contraction algebra for R_1 and Λ_2 contraction algebra for R_2

Donovan–Wemyss Conj (2013) $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda_1) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda_2) \overset{?}{\Longrightarrow} R_1 \cong R_2.$

August (2020) The contraction algebras of R form a single and complete derived equivalence class of algebras

Proof of the Donovan–Wemyss Conjecture

R isolated cDV singularity with crepant resolution

$\mathcal{D}_{\mathrm{sg}}(R)$ canonical DG enhancement of $\mathrm{D}_{\mathrm{sg}}(R)$

Hua–Keller (2018) The DG category $\mathcal{D}_{\mathrm{sg}}(R)$ determines R up to isomorphism:

$$\mathrm{HH}^0(\mathcal{D}_{\mathrm{sg}}(R)) \cong \frac{\mathbb{C}[\![u,v,x,t]\!]}{\left(f, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t}\right)}$$

is the Tyurina algebra of R, which determines R since $\dim R = 3$ is fixed **(Mather–Yau 1982)**.

Proof of the Donovan–Wemyss Conjecture II

R_1 and R_2 isolated cDV singularities with crepant resolutions

Λ_1 contraction algebra for R_1 and Λ_2 contraction algebra for R_2

Suppose that $\mathrm{D}^b(\operatorname{mod}\Lambda_1) \simeq \mathrm{D}^b(\operatorname{mod}\Lambda_2)$

Wemyss (2018) + August (2020)

$\exists\, T \in \mathrm{D}_{sg}(R_1)$ & $\exists\, S \in \mathrm{D}_{sg}(R_2)$ $2\mathbb{Z}$-cluster tilting such that

$$\operatorname{End}_{R_1}(T) \cong \Lambda_1 \cong \operatorname{End}_{R_2}(S)$$

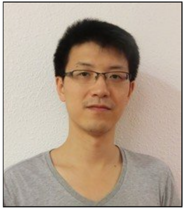
Triangulated Auslander–Iyama Correspondence $\implies$ $\operatorname{REnd}_{R_1}(T) \simeq \operatorname{REnd}_{R_2}(S)$

Will Donovan

Michael Wemyss

Jenny August

Bernhard Keller



Zheng Hua

Fernando Muro

Happy birthday Bernhard!

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