# The Triangulated Auslander-Iyama Correspondence

(joint work with Fernando Muro)

**Lecture 1** Statement of the Thm & applications

**Lecture 2** Enhanced $(d+2)$-angulated categories

**Lecture 3** $A_\infty$-structures on $d$-sparse graded algebras

# Lecture 1

## Standing assumptions

$\Bbbk$ : perfect field (e.g. $\operatorname{char} \Bbbk = 0$, $\Bbbk = \overline{\Bbbk}$ or $|\Bbbk| < \infty$)

All algebras are finite-dimensional (and basic, for simplicity)

All modules are right modules.

All categories are Hom-finite, additive & with split idempotents

# § The Triangulated Auslander Correspondence (after Muro)

$\mathcal{T}$: triangulated category. Suppose $\exists c \in \mathcal{T}$ s.t. $\mathrm{add}(c) = \mathcal{T}$.
basic

**Q** What can we say about $\Lambda := \mathcal{T}(c,c)$ ?

**Prop** (Freyd 1966, Heller 1968)

(1) $\Lambda$ is self-injective ($\Rightarrow$ $\underline{\mathrm{mod}}\,\Lambda$ is a triangulated category)

(2) $\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ induces $\Sigma : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda$ exact auto-equiv.

& $\quad \Sigma \cong \Omega_\Lambda^3$ as exact functors on $\underline{\mathrm{mod}}\,\Lambda$

k: perfect

**Def / Prop** (Green-Snashall-Solberg 2003, Hanihara 2020)

$A$: self-injective. Write $A^e := A \otimes A^{op}$. TFAE

(1) $\exists\, \sigma : A \xrightarrow{\sim} A$ algebra automorphism such that
$\Omega^n_{A^e}(A) \cong {}_1A_\sigma$ in $\underline{\mathrm{mod}}\,A^e$. (even in $\mathrm{mod}\,A^e$ if $A$ is connected & non-semisimple)

(2) $\exists\, u : A \xrightarrow{\sim} A$ algebra automorphism such that
$u^* \cong \Omega^n_A$ as exact functors on $\underline{\mathrm{mod}}\,A$.

We say that $A$ is twisted n-periodic (w.r.t. $\sigma$) if (1) holds.

**Coro** (Hanihara 2020) $\Lambda$ is twisted 3-periodic.

**Q** What about the converse ?

**Thm** (Amiot 2007) $\Lambda$ : twisted 3-periodic w.r.t. $\sigma : \Lambda \xrightarrow{\sim} \Lambda$

$\Longrightarrow$ $(\operatorname{proj}\Lambda,\ - \otimes_{\Lambda} {}_{\sigma}\Lambda_1)$ admits a triangulation.

**Q** Is Amiot's triangulation algebraic? Is it "unique"?

**Triangulated Auslander Correspondence** (Muro 2022)

There is a bijective correspondence between:

(1) Pairs $(\mathcal{T}, c)$ with $\mathcal{T}$ algebraic & $c \in \mathcal{T}$ st. $\operatorname{add}(c) = \mathcal{T}$ up to equivalence.

(2) Pairs $(\Lambda, \sigma)$ with $\Lambda$ twisted 3-periodic w.r.t. $\sigma$ $(\underbrace{\Omega^3_{\Lambda^e} \cong {}_1\Lambda_\sigma}_{\text{in } \operatorname{mod}\Lambda^e})$ up to equivalence.

Moreover, the triangulated categories in (1) admit a unique DG enhancement.

$$(\mathcal{T}, c) \sim (\mathcal{T}', c') \overset{\text{def}}{\Longleftrightarrow} \exists \quad \begin{array}{ccc} \mathcal{T} & \xrightarrow[\sim]{F} & \mathcal{T}' \\ \| & & \| \\ \operatorname{add}(c) & \xrightarrow[\sim]{F} & \operatorname{add}(c') \end{array} \quad F : \text{exact equiv.}$$

$$(\Lambda, \sigma) \sim (\Lambda', \sigma') \overset{\text{def}}{\Longleftrightarrow} \begin{tikzcd} \operatorname{proj}\Lambda \arrow[r, "F", "\sim"'] \arrow[d, "-\otimes_\Lambda {}_1\Lambda_\sigma"'] \arrow[dr, phantom, "G"] & \operatorname{proj}\Lambda' \arrow[d, "-\otimes_{\Lambda'} {}_1\Lambda'_{\sigma'}"] \\ \operatorname{proj}\Lambda \arrow[r, "F", "\sim"'] & \operatorname{proj}\Lambda' \end{tikzcd} \quad \text{up to natural iso.}$$

(in particular, $\Lambda \underset{\text{Morita}}{\sim} \Lambda'$)

**Rmk** Muro's proof shows that Amiot's triangulation is algebraic and unique up to exact equivalence.

## § The triangulated Auslander-Iyama Correspondence

Fix $d \geqslant 1$ an integer.

**Q** What do twisted $(d+2)$-periodic algebras correspond to?

**Def** (Iyama-Yoshino 2008, Geiß-Keller-Oppermann 2013)

$c \in \mathcal{T}$ is d-cluster tilting if

(1) $\operatorname{add}(c) = \{ x \in \mathcal{T} \mid \forall\, 0 < i < d \ \operatorname{Ext}^i_{\mathcal{T}}(x, c) = 0 \}$

Same if ambient category is abelian or exact.

$= \{ y \in \mathcal{T} \mid \forall\, 0 < i < d \ \operatorname{Ext}^i_{\mathcal{T}}(c, y) = 0 \}$

$c \in \mathcal{T}$ is $d\mathbb{Z}$-cluster tilting if $c$ is d-cluster tilting and

(2) $\forall i \notin d\mathbb{Z} \ \ \mathcal{T}(c, c[i]) = 0 \ \ \left( \Leftrightarrow \operatorname{add}(c)[d] = \operatorname{add}(c) \right)$

**Rmk** $c \in \mathcal{T}$ : 1-CT $(= 1\mathbb{Z}\text{-CT}) \Leftrightarrow \operatorname{add}(c) = \mathcal{T}$.

**Triangulated Auslander-Iyama Correspondence** (J-Muro)

There is a bijective correspondence between:

(1) Pairs $(\mathcal{T}, c)$ with $\mathcal{T}$ algebraic & $c \in \mathcal{T}$ : $d\mathbb{Z}$-CT up to equiv.

(2) Pairs $(\Lambda, \sigma)$ with $\Lambda$ twisted $(d+2)$-periodic w.r.t. $\sigma$ $\left( \Omega^{d+2}_{\Lambda^e} \cong {}_1\Lambda_\sigma \right)$ in $\operatorname{mod} \Lambda^e$

Moreover, the triangulated categories in (1) admit a unique DG enhancement.

**Rmk** The correspondence is given by $(\mathcal{T}, c) \longmapsto (\Lambda, \sigma)$ where $\Lambda := \mathcal{T}(c,c)$ & $\sigma : \Lambda \xrightarrow{\sim} \Lambda$ is an algebra automorphism such that the following diagram commutes up to natural isomorphism ($\mathscr{C} := \mathrm{add}(c)$):

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\mathcal{T}(c,-)"] \arrow[d, "{[d]}"'] & \mathrm{proj}\,\Lambda \arrow[d, "- \otimes_\Lambda {}_\sigma\Lambda_1"] \\ \mathscr{C} \arrow[r, "\mathcal{T}(c,-)"] & \mathrm{proj}\,\Lambda \end{tikzcd}$$

$\mathrm{Aut}(\Lambda)/\mathrm{Inn}(\Lambda)$
$\shortparallel$

If $\Lambda$ is connected & non-semisimple, then $[\sigma] \in \mathrm{Out}(\Lambda)$ is uniquely determined (equivalently, ${}_\sigma\Lambda_1$ is determined up to bimodule isomorphism) by $\Lambda$.

## § Recognition theorems

to exclude semisimple case

**Def** A: f.d. alg. with gl.dim $A = d$ is <u>d-representation finite</u> if $\exists M \in \mathrm{mod}\,A$ : d-CT.

A: d-RF $\rightsquigarrow$ $\mathscr{C}(A)$ : d-CY cluster category of A
(Amiot 2009, Guo 2011, Keller 2011)

$\Pi_{d+1}(A) := \bigoplus_{i \geq 0} \mathrm{Ext}^d_A(DA, A)^{\otimes_A i}$ : (d+1)-preproj algebra of A

**Thm** (Iyama-Oppermann 2013) A: d-RF alg. (connected)

$\mathscr{C}(A)$ $\underbrace{\text{admits a } d\mathbb{Z}\text{-CT object } c}_{①}$ $\underbrace{\text{with } \mathrm{End}(c) \cong \Pi_{d+1}(A)}_{②}$

**Coro** ① & ② characterise $\mathscr{C}(A)$ among algebraic tri. cat's.

**Def** (Herschend - Iyama 2011)

$(Q,W)$: quiver with potential is self-injective if the Jacobian algebra $J(Q,W)$ is finite-dimensional & self-injective.

**Thm** (Amiot 2009, Keller 2011, Iyama - Oppermann 2013)

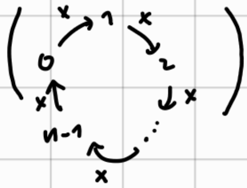
$$(Q,W): \text{ self-inj. QP (connected)} \Longrightarrow \mathcal{C}(Q,W) \overbrace{\text{ admits a } 2\mathbb{Z}\text{-CT object } c}^{③} \underbrace{\text{ with } \operatorname{End}(c) \cong J(Q,W)}_{④}$$

**Coro** ③ & ④ characterise $\mathcal{C}(Q,W)$ among algebraic tri. cat's.

**Thm** (J-Külshammer 2019) $A^{(d)}_{n-1,\ell}$ : self-inj. d-Nakayama alg.
using Darpö - Iyama 2020

$$\Longrightarrow \underline{\operatorname{mod}}\, A^{(d)}_{n-1,\ell} \underbrace{\text{ admits a } d\mathbb{Z}\text{-CT object } c}_{①} \text{ with } \underbrace{\operatorname{End}(c) \cong A^{(d+1)}_{n-1,\ell-1})}_{⑤}$$

**Coro** ① & ⑤ characterise $\underline{\operatorname{mod}}\, A^{(d)}_{n,\ell}$ among algebraic tri. cat's

**Ex** $A^{(1)}_{n-1,\ell} \cong k$  $/(x^\ell)$ self-inj Nakayama algebra

$$A^{(2)}_{0,\ell} = \Pi_2(\mathbb{A}_\ell) = k\left(1 \underset{a^*}{\overset{a}{\rightleftarrows}} 2 \underset{a^*}{\overset{a}{\rightleftarrows}} \cdots \underset{a^*}{\overset{a}{\rightleftarrows}} \ell\right) / (\Sigma aa^* - aa^*)$$

($\exists\, 2\mathbb{Z}$-CT object in $\underline{\operatorname{mod}}\, A$ due to Geiß - Leclerc - Schröer 2006)

$A^{(d+1)}_{0,\ell} = \Pi_{d+1}(A^{(d)}_\ell)$ : $(d+1)$-preproj. alg. of d-Auslander alg. of type $\vec{\mathbb{A}}_\ell$

($\exists\, (d+1)\mathbb{Z}$-CT in $\underline{\operatorname{mod}}\, A$ due to Iyama - Oppermann 2013)

**Rmk** Keller and Reiten (2008) established a Recognition Theorem for the d-Calabi-Yau cluster category of an acyclic quiver ($k = \bar{k}$)

**Rmk** Hanihara (2022) established a Recognition Theorem for orbit categories of the form "$\frac{1}{d-1}$-st root" of AR translation

$$\underbrace{D^b(\mathrm{mod}\, H) / \tau^{-1/(d-1)}[1]}_{\text{d-Calabi-Yau with a d-CT object (but not } d\mathbb{Z}\text{-CT)}}$$

where $H$: fin. dim. hereditary alg of infinite rep. type.

Hanihara's results also show uniqueness of enhancements.

**Rmk** Keller has announced a general Recognition Theorem for the (2-CY) cluster category of a Jacobi-finite quiver with potential. Keller's theorem requires an explicit assumption on the enhancements (the existence of a right CY structure in the sense of Kontsevich & Soibelman).

Note that Keller's thm deduces that the endomorphism algebra of the given 2-CT is a Jacobian algebra a posteriori.

**Rmk** (Twisted) periodic algebras are plentiful & include, among others,

- (Green-Snashall-Solberg 2003) self-injective algebras of finite type
- (Chan-Darpö-Iyama-Marczinzik 2020) trivial extensions of fractionally Calabi-Yau fin. dim. alg's of finite global dimension.

d-cluster tilting modules play a crucial role.

# Appendix

## § Fin.-dim. alg's are d-Calabi-Yau tilted $\forall\, d > 2$ (after Ladkani)

$\Bbbk = \overline{\Bbbk}$ : field & $A = \Bbbk Q / I$ : f.d. alg. $\quad J := \langle Q_1 \rangle \subseteq \Bbbk Q$ : arrow ideal

$R = \bigcup_{i,j \in Q_0} R_{i,j}$ , $R_{i,j} \subseteq e_i J^2 e_j$ : finite set of relations $j \rightsquigarrow i$
(repetitions & zero (0) are allowed!)

such that $I = \langle R \rangle$.

examples of non-equiv d-CY tri. cat. with d-CT object with isomorphic end. alg. but not $d\mathbb{Z}$-CT see below...

(Ladkani 2016) depends on R not on $I = \langle R \rangle$

$\rightsquigarrow \Gamma := \Gamma(Q, R, d)$ : DG algebra such that

(1) $\Gamma$ is homologically smooth : $\Gamma \in D^c(\Gamma \otimes \Gamma^{op})$

(2) $\Gamma$ is bimodule $(d+1)$-CY : $\mathbb{R}\mathrm{Hom}_{\Gamma^e}(\Gamma, \Gamma^e) = \Gamma[-(d+1)]$ in $D^c(\Gamma^e)$

(3) $\Gamma$ is connective : $H^{>0}(\Gamma) = 0$

(4) $H^0(\Gamma) \cong \Bbbk Q / \langle R \rangle = A$ (fin. dim)

$$\xRightarrow[\text{(Guo 2011)}]{\text{(Amiot 2009)}} \quad \mathcal{C} = \mathcal{C}(\Gamma) := D^c(\Gamma) / D^{fd}(\Gamma) \text{ is a } d\text{-CY tri. cat.}$$

$\Gamma$ : d-CT object $\Rightarrow$ $\sharp$ useful characterisation of end. alg's of d-CT object
with $\mathrm{End}_{\mathcal{C}}(\Gamma) = H^0(\Gamma) \cong A$

(5) $\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\Gamma, \Gamma[2-d]) \geq |R|$

$\therefore$ $\underbrace{\Gamma \in \mathcal{C}}_{\text{as in Ladkani's construction}}$ : $d\mathbb{Z}$-CT $\Longleftrightarrow$ $R = \emptyset$ & $Q_1 = \emptyset$ $\Longleftrightarrow$ $\mathcal{C}$ = d-cluster cat of $\Bbbk^{Q_0} = A$

beacause of (5) & $d > 2$

Since $A = \Bbbk Q$ must be $\underbrace{\text{fin. dim.}}_{\text{assumption}}$ & $\underbrace{\text{self. inj}}_{\text{by GKO's (d+2)-ang Freyd's Lemma}}$

# Lecture 2

## § (d+2)-angulated categories

$\mathcal{F}$: additive cat. & $\Sigma : \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ automorphism

"Def" (Geiß-Keller-Oppermann 2013) A (d+2)-angulation of $(\mathcal{F}, \Sigma)$ is a class of sequences $\pentagon = \{ \underbrace{x_{d+1} \to x_d \to \cdots \to x_1 \to x_0 \to \Sigma x_{d+1}}_{(d+2)\text{-angle}} \}$ that satisfy some axioms similar to those of triangulated categories.

$\rightsquigarrow (\mathcal{F}, \Sigma, \pentagon)$ : (d+2)-angulated category

Rmk 3-angulated category = triangulated category

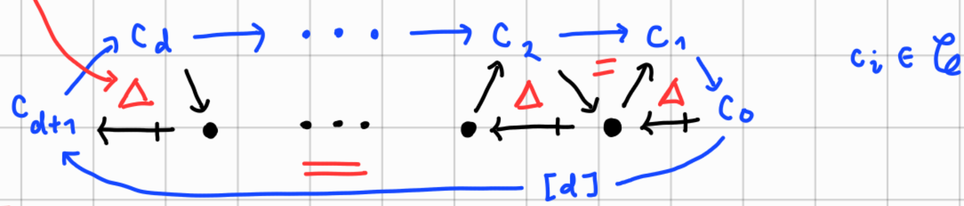
Thm (Geiß-Keller-Oppermann 2013) $\mathcal{C} \subseteq \mathcal{T}$ : $d\mathbb{Z}$-CT subcategory

$\Longrightarrow (\mathcal{C}, [d])$ has a (d+2)-angulation with (d+2)-angles

exact triangle

standard (d+2)-ang.

makes sense even if $\mathcal{C}$ is not $d\mathbb{Z}$-CT




$c_i \in \mathcal{C}$

Prop (Geiß-Keller-Oppermann 2013) $(\mathcal{F}, \Sigma, \pentagon)$ : (d+2)-angulated cat.

$\Longrightarrow$ mod $\mathcal{F}$ : Frobenius abelian & $\Sigma \cong \Omega^{d+2}_{\mathcal{F}}$ as exact functors on mod $\mathcal{F}$

Coro $(\mathcal{F}, \Sigma, \pentagon)$ : $(d+2)$-ang. cat. Suppose $\exists\, x \in \mathcal{F}$ st. $\mathrm{add}(x) = \mathcal{F}$.

$\Longrightarrow$ $\mathcal{F}(x,x)$ is self-inj. & twisted $(d+2)$-periodic.

Coro (Chan - Darpö - Iyama - Marczinzik 2020)

$c \in \mathcal{T}$ : $d\mathbb{Z}$-CT $\Rightarrow$ $\mathcal{T}(c,c)$ is self-inj. & twisted $(d+2)$-periodic.

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Q What about the converse?

$\Lambda$ : twisted $(d+2)$-periodic w.r.t. $\sigma : \Lambda \xrightarrow{\sim} \Lambda$ $\Big( \underset{\text{in mod } \Lambda^e}{\Omega^{d+2}_{\Lambda^e}(\Lambda) \cong {}_1\Lambda_\sigma} \Big)$

Choose $\delta : 0 \to {}_1\Lambda_\sigma \to \underbrace{P_{d+1} \to \cdots \to P_1 \to P_0}_{\text{projective (-injective)}} \to \Lambda \to 0$ : ex. seq. of $\Lambda$-bimod's

Def (Amiot 2007 $d=1$, Lin 2019) $\Sigma := - \otimes_\Lambda {}_\sigma\Lambda_1 : \mathrm{proj}\,\Lambda \xrightarrow{\sim} \mathrm{proj}\,\Lambda$

$X_{d+2} \xrightarrow{f_{d+2}} X_{d+1} \xrightarrow{f_{d+1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} \Sigma X_{d+2}$ is a $\delta$-exact $(d+2)$-angle in $\mathrm{proj}\,\Lambda$ if

(1) $X_{d+2} \to X_{d+1} \to \cdots \to X_1 \xrightarrow{f_1} \Sigma X_{d+2} \xrightarrow{\Sigma f_{d+2}} \Sigma X_{d+1}$ is exact

(2) $N := \mathrm{coker}\, f_1 \in \mathrm{mod}\,\Lambda$. The exact sequences

(i) $0 \to \Sigma^{-1} N \to X_{d+2} \to X_{d+1} \to \cdots \to X_1 \to N \to 0$ (does not depend on $\delta$)

(ii) $N \underset{\Lambda}{\otimes} \left( 0 \to {}_1\Lambda_\sigma \to P_{d+1} \to \cdots \to P_1 \to P_0 \to \Lambda \right)$

are equivalent in $\mathrm{Ext}^{d+2}_\Lambda(N, \Sigma^{-1} N)$.

**Thm** (Amiot 2007 $d=1$, Lin 2019)

$\Lambda$ : twisted $(d+2)$-periodic w.r.t. $\sigma : \Lambda \xrightarrow{\sim} \Lambda$

$\Sigma : - \otimes_{\Lambda} {}_{\sigma}\Lambda_1 : \operatorname{proj} \Lambda \xrightarrow{\sim} \operatorname{proj} \Lambda$

$\pentagon_\delta$ : $\delta$-exact $(d+2)$-angles in $\operatorname{proj}\Lambda$ (J-Muro : independent of $\delta$ up to equiv.)

$\Longrightarrow$ $(\operatorname{proj}\Lambda, \Sigma, \pentagon_\delta)$ : $(d+2)$-angulated category.

## § Enhanced $(d+2)$-angulated categories

$\mathcal{A}$ : (small) dg category

$\forall\, x, y \in \mathcal{A} \rightsquigarrow \mathcal{A}(x,y) \in C(\operatorname{Mod} k)$

graded Leibniz rule

$\mathcal{A}(y,z) \otimes \mathcal{A}(x,y) \longrightarrow \mathcal{A}(x,z)$ chain map $\left( d(gf) = d(g)f + (-1)^{|g|} g \cdot d(f) \right)$

$\rightsquigarrow$ $H^\bullet(\mathcal{A})$ graded cat. with $H^\bullet(\mathcal{A})(x,y) = H^\bullet(\mathcal{A}(x,y))$

$H^0(\mathcal{A})$ ordinary cat. with $H^0(\mathcal{A})(x,y) = H^0(\mathcal{A}(x,y))$

$\rightsquigarrow$ $D(\mathcal{A})$ : derived cat (compactly gen. tri. cat., Keller 1994)

$H^0(\mathcal{A}) \xhookrightarrow{\text{can}} D(\mathcal{A})$, $x \longmapsto h_x := \mathcal{A}(-,x)$ "free DG $\mathcal{A}$-module"

$\searrow$ $\cup\!\!\!/\!/$

$D^c(\mathcal{A}) := \operatorname{thick}(h_x \mid x \in \mathcal{A})$ perfect derived cat.

closure under $[\pm 1]$, cones & direct summands

(tri. cat with split idempotents)

**Def** (Bondal-Kapranov 1990)

$\mathcal{A}$ is Karoubian pre-triangulated if $\text{can} : H^0(\mathcal{A}) \hookrightarrow D^c(\mathcal{A})$ is an equivalence

**Def** (Bondal-Kapranov 1990) $\mathcal{T}$: tri. cat (with split idempotents)

An enhancement of $\mathcal{T}$ is a Karoubian pre-tri. DG cat $\mathcal{A}$ such that $\mathcal{T} \simeq H^0(\mathcal{A})$ as triangulated categories.

**Def** $\mathcal{T}$: tri. cat. $\mathcal{C} \subseteq \mathcal{T}$ is $d\mathbb{Z}$-rigid if $\forall i \notin d\mathbb{Z}$ $\mathcal{T}(\mathcal{C}, \mathcal{C}[i]) = 0$

**Def / Thm** (J-Muro) $H^0(\mathcal{A})$: Hom-finite, $H^0(\mathcal{A}) \xrightarrow[\text{can}]{\sim} \mathcal{C} \subseteq D^c(\mathcal{A})$. TFAE

(1) $\mathcal{C} \subseteq D^c(\mathcal{A})$ is $d\mathbb{Z}$-cluster tilting

(GKO 2013) $\Downarrow$

(2) (i) $\mathcal{C}$ is $d\mathbb{Z}$-rigid & $\mathcal{C}[d] = \mathcal{C}$

(ii) The standard $(d+2)$-angles in $\mathcal{C}$ form a $(d+2)$-angulation of $(\mathcal{C}, [i])$

If these cond. hold, $\mathcal{A}$ is Karoubian pre-$(d+2)$-angulated

**Def** (J-Muro) $(\mathcal{F}, \Sigma, \pentagon)$: $(d+2)$-ang. cat. (with split idempotents)

An enhancement of $\mathcal{F}$ is a Karoubian pre-$(d+2)$-angulated DG cat such that $H^0(\mathcal{A}) \simeq \mathcal{F}$ as $(d+2)$-angulated categories.

**Def** $F: \mathcal{A} \to \mathcal{B}$ DG functor is a quasi-equivalence if the induced graded functor $H^*(F): H^*(\mathcal{A}) \to H^*(\mathcal{B})$ is an equivalence.

**Def** $(\mathcal{F}, \Sigma, \pentagon)$: $(d+2)$-ang. cat. (with split idempotents)

$\mathcal{F}$ has a unique enhancement if it has an enhacement and any two enhancements of $\mathcal{F}$ are quasi-equivalent (via zig-zag of quasi-eq's).

# § Enhanced $(d+2)$-angulated categories of finite type

$(\Lambda, \sigma)$ with $\Lambda$ twisted $(d+2)$-periodic w.r.t. $\sigma : \Lambda \xrightarrow{\sim} \Lambda$

$\Sigma := - \otimes_\Lambda {}_\sigma\Lambda_1 : \operatorname{proj}\Lambda \xrightarrow{\sim} \operatorname{proj}\Lambda$

$\pentagon_\delta$ : class of $\delta$-exact $(d+2)$-angles

**Thm** (J-Muro) The AL $(d+2)$-angulation $(\mathcal{F}, \Sigma, \pentagon_\delta)$ admits a unique enhancement.

$\Longrightarrow$ Triangulated Auslander-Iyama Correspondence (surjectivity)

$(\Lambda, \sigma)$ with $\Lambda$ twisted $(d+2)$-periodic w.r.t. $\sigma : \Lambda \xrightarrow{\sim} \Lambda$

$\Sigma := - \otimes_\Lambda {}_\sigma\Lambda_1 : \operatorname{proj}\Lambda \xrightarrow{\sim} \operatorname{proj}\Lambda$

$\pentagon_\delta$ : class of $\delta$-exact $(d+2)$-angles

$\overset{\text{Thm}}{\Longrightarrow}$ $\exists\, \mathcal{A}$ : enhancement of $(\operatorname{proj}\Lambda, \Sigma, \pentagon_\delta)$

$$\begin{tikzcd}
\operatorname{proj}\Lambda \simeq H^0(\mathcal{A}) \arrow[r, hook, "d\mathbb{Z}\text{-CT}"] & D^c(\mathcal{A}) =: \mathcal{T} \\
\Lambda \arrow[r, mapsto] & c
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{proj}\Lambda \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\Sigma"'] & H^0(\mathcal{A}) \arrow[d, "{[d]}"] \\
\operatorname{proj}\Lambda \arrow[r, leftrightarrow, "\sim"] & H^0(\mathcal{A})
\end{tikzcd}$$

$\therefore\ (\mathcal{T}, c) \longmapsto (\Lambda, \sigma)$ ∎

**Key problem** $(\Lambda, \sigma)$ with $\Lambda$ twisted $(d+2)$-periodic w.r.t. $\sigma : \Lambda \xrightarrow{\sim} \Lambda$

$\mathcal{A}$ & $\mathcal{B}$ : Karoubian pre-$(d+2)$-angulated DG cat's such that

$$(H^0(\mathcal{A}), [d]) \simeq (\operatorname{proj}\Lambda, \Sigma) \simeq (H^0(\mathcal{B}), [d])$$

Have two induced $(d+2)$-angulations on $(\operatorname{proj}\Lambda, \Sigma)$. Why do they agree?

# Lecture 3

## § Amiot-Lin $(d+2)$-angulations, revisited

$\Lambda$ : twisted $(d+2)$-periodic w.r.t. to $\sigma : \Lambda \xrightarrow{\sim} \Lambda$

$\Sigma := - \otimes_{\Lambda} {}_{\sigma}\Lambda_1 : \operatorname{proj} \Lambda \xrightarrow{\sim} \operatorname{proj} \Lambda$

$\delta : 0 \to {}_1\Lambda_\sigma \to \underbrace{P_{d+1} \to \cdots \to P_1 \to P_0}_{\text{projective}} \to \Lambda \to 0$ : ex. seq of $\Lambda$-bimod's

$\rightsquigarrow$ $\pentagon_\delta$ : class of $\delta$-exact $(d+2)$-angles in $\operatorname{proj} \Lambda$

By definition, $[\delta] \in \operatorname{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma)$

Recall $\quad M \in \operatorname{Mod} \Lambda^e \rightsquigarrow HH^\bullet(\Lambda, M) := \operatorname{Ext}^\bullet_{\Lambda^e}(\Lambda, M)$ Hochschild cohomology of $\Lambda$ with coeff. in $M$

$\therefore \quad [\delta] \in \operatorname{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma) = HH^{d+2}(\Lambda, {}_1\Lambda_\sigma)$

for coeff in diag. bimod.

Upshot $\quad$ Hochschild cohomology has a rich algebraic structure (Gerstenhaber algebra) as well as a graded variant.

$(\operatorname{proj} \Lambda, \Sigma) \rightsquigarrow (\operatorname{proj} \Lambda)^\Sigma$ : graded category with

- objects $= \operatorname{proj} \Lambda$
- morphisms $\operatorname{Hom}^j_\Sigma(P, Q) = \begin{cases} \operatorname{Hom}_\Lambda(P, \Sigma^{j/d} Q) & j \in d\mathbb{Z} \\ 0 & j \notin d\mathbb{Z} \end{cases}$

$\Lambda(\sigma, d) := \operatorname{Hom}^\bullet_\Sigma(\Lambda, \Lambda)$ with $\Lambda(\sigma, d)^{di} \cong {}_{\sigma^i}\Lambda_1 \quad (\deg 0 = \Lambda)$

graded $\Lambda$-module

$$\rightsquigarrow \mathrm{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma,d)) := \mathrm{Ext}^{\bullet,*}_{\Lambda^e}(\Lambda, \Lambda(\sigma,d))$$ bigraded algebra

$$\mathrm{H}^{p,q}(\Lambda, \Lambda(\sigma,d)) = \mathrm{HH}^{p}(\Lambda, \Lambda(\sigma,d)^q)$$

$$\mathrm{HH}^{d+2,-d}(\Lambda, \Lambda(\sigma,d)) = \mathrm{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma)$$

$$\rightsquigarrow \underline{\mathrm{HH}}^{\bullet,*}(\Lambda, \Lambda(\sigma,d)) := \underline{\mathrm{Ext}}^{\bullet,*}_{\Lambda^e}(\Lambda, \Lambda(\sigma,d))$$ Hochschild-Tate cohomology
also bigraded algebra

$$\mathrm{HH}^{>0,*}(\Lambda, \Lambda(\sigma,d)) \xrightarrow{\sim} \underline{\mathrm{HH}}^{>0,*}(\Lambda, \Lambda(\sigma,d))$$

**Prop** (Muro 2022) $\eta : 0 \to {}_1\Lambda_\sigma \to X \to \underbrace{P_d \to \cdots \to P_1 \to P_0}_{\text{projective}} \to \Lambda \to 0$ : ex. seq. of $\Lambda$-bimod's

$[\eta] \in \underline{\mathrm{HH}}^{\bullet,*}(\Lambda, \Lambda(\sigma,d))$ : unit (w.r.t. cup product) $\Longleftrightarrow$ $X$ is projective

$\rightsquigarrow [\delta] \in \underline{\mathrm{HH}}^{\bullet,*}(\Lambda, \Lambda(\sigma,d))$ is a unit (a key property!)

Slogan: Amiot-Lin $(d+2)$-angulations are determined by units in $\underline{\mathrm{HH}}^{d+2,-d}(\Lambda, \Lambda(\sigma,d)) = \underline{\mathrm{Ext}}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma)$

## § $A_\infty$-structures on $d$-sparse graded algebras

$\mathcal{A}$ : Karoubian pre-$(d+2)$-angulated category

Suppose that

- $\mathcal{F} := H^0(\mathcal{A})$ is Hom-finite
- $\exists\, c \in \mathcal{F}$ s.t. $\mathrm{add}(c) = \mathcal{F}$

$\Lambda := \mathcal{F}(c,c)$ twisted $(d+2)$-per.

$$\Rightarrow \begin{tikzcd} \mathcal{F} \arrow[r, "\mathcal{F}(c,-)", "\sim"'] \arrow[d, "{[d]}"'] \arrow[dr, phantom, "\circlearrowleft"] & \operatorname{proj}\Lambda \arrow[d, "- \otimes_\Lambda {}_\sigma\Lambda_1"] \\ \mathcal{F} \arrow[r, "\mathcal{F}(c,-)", "\sim"'] & \operatorname{proj}\Lambda \end{tikzcd}$$

some $\sigma$

$H^\bullet(\mathcal{A})(c,c) \cong \Lambda(\sigma,d)$ inherits minimal ($m_1=0$) $A_\infty$-structure (Kadeishvili 1982)

For $n \geq 3$, $m_n : \Lambda(\sigma,d)^{\otimes n} \longrightarrow \Lambda(\sigma,d)$ $\quad |m_n| = 2-n$

$i \geq 1 \quad m_{i+2} \quad\quad |m_{i+2}| = -i$

$\Lambda(\sigma,d)$ is d-sparse: $\forall i \notin d\mathbb{Z}$ $\Lambda(\sigma,d)^i = 0$.

$\therefore \underbrace{m_{i+2} = 0 \quad \forall i \notin d\mathbb{Z}}_{\circledast} \overset{\text{only}}{\underset{\text{have}}{\rightsquigarrow}} m_{d+2}, m_{2d+2}, m_{3d+2}, \ldots$

Notice $m_{i+2} \in C^{i+2,-i}(\Lambda(\sigma,d), \Lambda(\sigma,d))$: Hochschild complex $C^{\bullet,*}$

Moreover $\partial_{\text{Hoch}}(m_{d+2}) = 0$ (Lefèvre-Hasegawa 2003, using $\circledast$)

Universal Massey product (of length $d+2$) $\{m_{d+2}\} \in HH^{d+2,-d}(\Lambda(\sigma,d), \Lambda(\sigma,d))$

independent of min. $A_\infty$-model

$$\big\downarrow j^* \quad (j : \Lambda \xrightarrow[\deg 0]{} \Lambda(\sigma,d))$$

Restricted universal Massey product $\quad j^*\{m_{d+2}\} \in HH^{d+2,-d}(\Lambda, \Lambda(\sigma,d))$

$$\| $$

$$\mathrm{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_\sigma)$$

$\therefore$ $j^*\{m_{d+2}\}$ is represented by an extension of $\Lambda$-bimod's

$$\delta : 0 \to {}_1\Lambda_\sigma \to X \to \underbrace{P_d \to \cdots \to P_1 \to P_0}_{\text{projective}} \to \Lambda \to 0$$

k-perfect

**Prop** (J-Muro) $j^*\{m_{d+2}\} \in \underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma,d))$ is a unit (i.e $X$ is proj.)

Moreover std $(d+2)$-angles $\simeq$ $\delta$-exact $(d+2)$-angles

in $(H^0(\mathcal{A}), [d])$ $\simeq$ in $(\operatorname{proj}\Lambda, \Sigma)$

$\mathcal{A}$ : small DG cat & $H^0(\mathcal{A}) \xrightarrow[\text{can}]{\sim} \mathcal{C} \subseteq D^c(\mathcal{A})$ : Hom-finite

Suppose that $\mathcal{C} \subseteq D^c(\mathcal{A})$ is $d\mathbb{Z}$-rigid, $\mathcal{C}[d] = \mathcal{C}$, and closed under finite direct sums & direct summands.

Moreover, suppose $\exists c \in \mathcal{C}$ s.t. $\mathrm{add}(c) = \mathcal{C}$. Set $\Lambda := \mathcal{C}(c,c)$ and $\sigma : \Lambda \xrightarrow{\sim} \Lambda$ an automorphism s.t.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{C}(c,-)", "\sim"'] \arrow[d, "{[d]}"'] & \mathrm{proj}\,\Lambda \arrow[d, "- \otimes_{\Lambda} {}_{\sigma}\Lambda_1"] \\ \mathcal{C} \arrow[r, "\sim"'] & \mathrm{proj}\,\Lambda \end{tikzcd}$$

**Thm** (J-Muro) TFAE

(1) $\mathcal{C} \subseteq D^c(\mathcal{A})$ is $d\mathbb{Z}$-CT

(2) $j^*\{m_{d+2}\} \in HH^{\bullet,*}(\Lambda, \Lambda(\sigma,d))$ is a unit

**Thm** (J-Muro) $\Lambda$ : twisted $(d+2)$-periodic w.r.t. $\sigma : \Lambda \xrightarrow{\sim} \Lambda$

$\Longrightarrow$ $\exists$ minimal $A_\infty$-alg. structure $(\Lambda(\sigma,d), m_{d+2}, m_{2d+2}, m_{3d+2}, \dots)$ such that $j^*\{m_{d+2}\} \in \underline{HH}^{\bullet,*}(\Lambda, \Lambda(\sigma,d))$ is a unit, i.e

$$j^*\{m_{d+2}\} \in HH^{d+2,-d}(\Lambda, \Lambda(\sigma,d)) = \mathrm{Ext}^{d+2}_{\Lambda^e}(\Lambda, {}_1\Lambda_{\sigma})$$

can be represented by an exact sequence of $\Lambda$-bimod's

$$0 \to {}_1\Lambda_{\sigma} \to \underbrace{P_{d+1} \to \cdots \to P_1 \to P_0}_{\text{projective}} \to \Lambda \to 0$$

$\Lambda$ is FD alg is used here → Moreover, any two min. $A_\infty$-algebras as above are quasi-isomorphic

$\iff$ Existence & uniqueness of enhanced $(d+2)$-ang. structures in finite type

$\Longrightarrow$ Triangulated Auslander–Iyama Correspondence (injectivity)

$(\mathcal{T}_i, c_i)$ with $\mathcal{T}_i$ alg. tri. cat & $c_i \in \mathcal{T}_i : d\mathbb{Z}$-CT $(i=1,2)$

Set $\Lambda_i := \mathcal{T}_i(c_i, c_i)$ & $\sigma_i : \Lambda_i \xrightarrow{\sim} \Lambda_i$ corresp. alg. automorphism

Suppose $(\Lambda_1, \sigma_1) \sim (\Lambda_2, \sigma_2)$.

$$\begin{array}{llll} \mathcal{B}_i : \text{pre-triang. DG cat st.} & H^0(\mathcal{B}_i) & \simeq \mathcal{T}_i & \text{as tri. cat's} \\ \cup\!| & \cup\!| & \cup\!| & \\ \mathcal{A}_i : \text{full DG subcat. spanned by} & \mathcal{C}_i & \simeq \operatorname{add}(c_i) & \end{array}$$

$\rightsquigarrow$ $(H^\bullet(\mathcal{A}_i), m^{(i)}_{d+2}, m^{(i)}_{2d+2}, m^{(i)}_{3d+3}, \dots)$ min $A_\infty$-structure

$$j^*\{m^{(i)}_{d+2}\} \in \underline{HH}^{\bullet,*}(H^0(\mathcal{A}_i), H^\bullet(\mathcal{A}_i)) \text{ is a unit}$$
$$\cong$$
$$\underline{HH}^{\bullet,*}(\Lambda_i, \Lambda_i(\sigma_i, d))$$

Thm $\overset{(1)}{\Longrightarrow}$ $(H^\bullet(\mathcal{A}_1), m^{(1)}_{*d+2}) \underset{\text{quasi-eq}}{\overset{A_\infty}{\sim}} (H^\bullet(\mathcal{A}_2), m^{(2)}_{*d+2})$

$\overset{(2)}{\Longrightarrow}$ $\mathcal{A}_1 \underset{\text{quasi-eq}}{\overset{DG}{\sim}} \mathcal{A}_2$ $\overset{(3)}{\Longrightarrow}$ $\underbrace{\mathcal{B}_1 \underset{\text{quasi-eq}}{\overset{DG}{\sim}} \mathcal{B}_2}_{\text{uniqueness of enhancements}}$

$$\Longrightarrow \underbrace{\begin{array}{ccc} \mathcal{T}_1 & & \mathcal{T}_2 \\ \cong & & \cong \\ H^0(\mathcal{B}_1) & \underset{\text{eq}}{\overset{\Delta}{\simeq}} & H^0(\mathcal{B}_2) \end{array}}_{\text{injectivity of the correspondence}}$$

(1) $\mathcal{A}_i$ is htpy Karoubian envelope of $((\Lambda_i, \sigma_i), m^{(i)}_{d+2}, \dots)$

(2) Rectification

(3) Morita theory (thick($d$-CT) $= \mathcal{T}$)

**Thank you for your attention!**

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