The triangulated Auslander - Iyama Correspondence I

(joint work with Fernando Muro)

§ Motivation

$\mathcal{T}$: (ess. small) tri. cat / k : field
- idempotent complete
- finite dimensional Hom-spaces

Fix : $G \in \mathcal{T}$ generator
($\mathrm{thick}(G) = \mathcal{T}$)

Question What is needed to reconstruct $\mathcal{T}$ from G as a tri. cat. ?

In general, neither $\mathcal{T}(G,G)$ nor $\mathcal{T}(G,G)^\bullet \cong \bigoplus_{i\in\mathbb{Z}} \mathcal{T}(G,G[i])$ suffice.

perfect der. cat.

$\mathcal{T}$: algebraic $\overset{\text{Keller}}{\underset{1994}{\Longrightarrow}}$ $\exists$ A : DG alg. s.t. $\mathcal{T} \underset{\Delta}{\simeq} D^c(A)$ & $H^\bullet(A) \cong \mathcal{T}(G,G)^\bullet$

Question Under which conditions is such A unique up to quasi-iso ?

Pseudo Thm A, B : DG alg's. $H^\bullet(A) \cong H^\bullet(B)$ + (?) $\Longrightarrow$ $A \underset{qi}{\sim} B$

Pseudo Coro $D^c(A) \simeq D^c(B)$ as enhanced triangulated categories

Thm (Kadeishvili 1988) $\forall\, p>0$ $HH^{p+2,-p}(H^\bullet(A)) = 0 \Longrightarrow A \underset{qi}{\simeq} H^\bullet(A)$ $\partial = 0$

good start, but too restrictive

F. Muro's Talk Less restrictive variant of Kadeishvili's Theorem

This talk Focus on properties of $A \in D^c(A)$

§ Algebraic triangulated categories of finite type (after Muro)

Def $\mathcal{T}$ is algebraic of finite type if $\exists$ A: DG algebra such that:

- $\mathcal{T} \underset{\Delta}{\simeq} D^c(A)$ as triangulated categories ($\Longrightarrow$ algebraic in sense of Keller)

- $\Lambda := H^0(A) \cong \mathrm{Hom}_{D(A)}(A, A)$ is a basic finite-dimensional algebra

- $\mathrm{add}(A) = D^c(A)$ ($\Longrightarrow \mathrm{Hom}_{D(A)}(A, -) : D^c(A) \xrightarrow{\sim} \mathrm{proj}(\Lambda)$ as additive cat's)

  ↑ closure under finite direct sums & direct summands

Fix A: DG algebra as above as well as $\varphi \in \mathrm{Hom}_{D(A)}(A, A[1]) \cong H^1(A)$ invertible

Define $\sigma = \sigma_\varphi \in \mathrm{Aut}(\Lambda)$ by $a \mapsto \varphi^{-1} a \varphi$. There is an iso of graded algebras

$$\boxed{H^*(A) \cong \Lambda(\sigma) := \bigoplus_{i \in \mathbb{Z}} {}_{\sigma^i}\Lambda_1, \quad a \cdot b := \sigma^j(a) b \text{ if } b \in \Lambda(\sigma)^j}$$

Prop The following statements hold: ($\Lambda^e := \Lambda \otimes \Lambda^{op}$: enveloping algebra)

(Freyd 1966) Λ is a Frobenius algebra

(Heller 1968) $\Omega^3_\Lambda \cong \sigma^*$ as exact functors on $\underline{\mathrm{mod}}(\Lambda)$

(Green-Snashall-Solberg 2003, Hanihara 2020) $\Bbbk$: perfect $\Longrightarrow \underbrace{\Omega^3_{\Lambda^e}(\Lambda) \overset{\text{stable iso}}{\simeq} {}_1\Lambda_\sigma}_{\Lambda \text{ is twisted 3-periodic w.r.t. } \sigma}$

Question Is A determined up to quasi-isomorphism by (Λ, σ)?

Remarkably YES!

Triangulated Auslander Correspondence (Muro 2022) $\Bbbk$: perfect field

There are bijective correspondences between the following:

A

- ⊛ DG algebras A such that $H^0(A)$ is fin. dim & $\operatorname{add}(A) = D^c(A)$ up to quasi-isomorphism
- ⊛ Pairs $(\mathcal{T} \ni \overset{\text{basic}}{c})$ where $\mathcal{T}$ is an algebraic tri. cat. of finite type & $\operatorname{add}(c) = \mathcal{T}$ up to equivalence of triangulated categories

$(H^0(A), \sigma)$ induced by $A \xrightarrow{\sim} A[1]$

- ⊛ Pairs $(\Lambda, \sigma \in \operatorname{Aut}(\Lambda))$ where Λ basic twisted 3-periodic w.r.t. σ up to algebra isomorphisms that preserve $[\sigma] \in \operatorname{Out}(\Lambda)$.

The correspondences are given by $A \longmapsto (D^c(A) \ni A)$ and

$$(\mathcal{T} \ni c) \longmapsto (\mathcal{T}(c,c), \sigma) \quad \text{where} \quad [1] \circlearrowright \mathcal{T} \xrightarrow[\sim]{\mathcal{T}(c,-)} \operatorname{proj} \mathcal{T}(c,c) \circlearrowleft - \otimes_\Lambda {}_\sigma\Lambda_1$$

Warning In general $A \not\simeq_{qi} H^*(A)$ for A as above.

Coro (M 2022) $\mathcal{T}$: alg. tri. cat. of finite type. TFSH

- ⊛ $\mathcal{T}$ admits a unique enhancement
- ⊛ $\mathcal{S}$: alg. tri. cat. such that there exist an equivalence of additive categories $F : \mathcal{T} \xrightarrow{\sim} \mathcal{S}$ & $F \circ [1] \cong [1] \circ F$

$\Longrightarrow$ $\mathcal{T} \simeq \mathcal{S}$ as triangulated categories.

Question What about pairs $(\Lambda, \sigma \in \operatorname{Aut}(\Lambda))$ such that Λ is a Frobenius algebra and $\underbrace{\Omega^{d+2}_{\Lambda^e}(\Lambda) \simeq {}_1\Lambda_\sigma}_{\Lambda \text{ is twisted } (d+2)\text{-periodic w.r.t. } \sigma}$?

§ A higher-dimensional generalisation $d \geq 1$

A : DG algebra such that $\Lambda := H^0(A)$ is a (basic) finite dimensional algebra

Want "d-dimensional analogue" of $\mathrm{add}(A) = D^c(A)$

Def / Thm (Iyama-Yoshino 2008, Geiß-Keller-Oppermann 2013, Beligiannis 2015) $d \geq 1$

A basic obj. $c \in \mathcal{T}$ is $d\mathbb{Z}$-cluster tilting if

(GKO)

- $\boxed{\exists\, c \xrightarrow{\sim} c[d]}$ & $\forall i \notin d\mathbb{Z}$ $\mathcal{T}(c, c[i]) = 0$
- $\mathrm{add}(c) * \mathrm{add}(c[1]) * \dots * \mathrm{add}(c[d-1]) = \mathcal{T}$

d-cluster tilting (IY)

Equivalently: $\exists\, c \xrightarrow{\sim} c[d]$ & $\boxed{\begin{array}{l} \mathrm{add}(c) = \{ x \in \mathcal{T} \mid \forall\, 0 < i < d \quad \mathcal{T}(x, c[i]) = 0 \} \\ \mathrm{add}(c) = \{ y \in \mathcal{T} \mid \forall\, 0 < i < d \quad \mathcal{T}(c, y[i]) = 0 \} \end{array}}$

Rmk $c \in \mathcal{T}$ is $1\mathbb{Z}$-cluster tilting $\Leftrightarrow$ $\mathrm{add}(c) = \mathcal{T}$

Fix A : DG algebra s.t. $A \in D^c(A)$ is $d\mathbb{Z}$-CT and $\varphi \in \mathrm{Hom}_{D(A)}(A, A[d]) \cong H^d(A)$ invertible

$$\rightsquigarrow H^*(A) \cong \Lambda(\sigma, d) := \bigoplus_{di \in d\mathbb{Z}} {}_{\sigma^i}\Lambda_1 \quad \text{d-sparse graded algebra} \quad (\Lambda := H^0(A))$$

d-sparse ↳ zero in degrees $\notin d\mathbb{Z}$

Prop (GKO 2013, GSS 2003, H 2022)

GKO:
- ⊛ $\Lambda := H^0(A)$ is a Frobenius algebra
- ⊛ $\Omega_{\Lambda}^{d+2} \cong \sigma^*$ as exact functors on $\underline{\mathrm{mod}}(\Lambda)$

GSS + H:
- ⊛ k: perfect $\Longrightarrow$ $\Omega_{\Lambda^e}^{d+2}(\Lambda) \cong {}_1\Lambda_\sigma$, i.e. Λ is twisted $(d+2)$-periodic w.r.t. σ

Triangulated Auslander-Iyama Correspondence (J-Muro) k: perfect field

There are bijective correspondences between the following:

A

⊛ DG algebras A such that $H^0(A)$ is fin. dim. & $A \in D^c(A)$ is $d\mathbb{Z}$-CT up to quasi-isomorphism

mind our assumptions on $\mathcal{T}$'s

⊛ Pairs $(\mathcal{T} \ni c)$ where $\mathcal{T}$ is an algebraic tri. cat. & $c \in \mathcal{T}$ is $d\mathbb{Z}$-cluster tilting up to equivalence of triangulated categories preserving $\mathrm{add}(c)$.

$(H^0(A), \sigma)$ ⊛ Pairs $(\Lambda, \sigma \in \mathrm{Aut}(\Lambda))$ where Λ is twisted $(d+2)$-periodic w.r.t. σ up to algebra isomorphisms that preserve $[\sigma] \in \mathrm{Out}(\Lambda)$.

induced by $A \xrightarrow{\sim} A[d]$

The correspondences are given by $A \longmapsto (D^c(A) \ni A)$ and $(\mathcal{T}, c) \longmapsto (\mathcal{T}(c,c), \sigma)$ where $[d] \circlearrowright \mathrm{add}(c) \xrightarrow[\sim]{\mathcal{T}(c,-)} \mathrm{proj}\, \mathcal{T}(c,c) \circlearrowleft - \otimes_{\Lambda} {}_{\sigma}\Lambda_1$

Coro (JM) $\mathcal{T}$: alg. tri. cat. & $c \in \mathcal{T}$: $d\mathbb{Z}$-cluster tilting object.

⊛ $\mathcal{T}$ admits a unique enhancement

⊛ $\mathcal{S}$: alg. tri. cat. & $c' \in \mathcal{S}$: $d\mathbb{Z}$-cluster tilting object such that there exists

$\mathcal{T} \supseteq \mathrm{add}(c) \xrightarrow{F} \mathrm{add}(c') \subseteq \mathcal{S}$ equivalence of additive categories

and $F \circ [d] \cong [d] \circ F$ as additive functors.

$\Longrightarrow$ $\mathcal{T} \simeq \mathcal{S}$ as triangulated categories.

Applications The following tri. categories satisfy the assumptions of the corollary:

(d = 1) mod(A), A: self-inj. of finite rep. type
[Muro 2020]
Gproj(A), A: fin. dim. Iwanaga-Gorenstein algebra of finite GP type
$\mathcal{C}(Q)$: BMRRT cluster category of Dynkin quiver Q 2006
...

(d = 2) mod Π, Π: preprojective alg. of type $\mathbb{A}_n$ Geiß-Leclerc-Schröer 2006
$\mathcal{C}(Q,W)$: Amiot cluster category of self-inj. quiver with potential (Q,W) 2009
$\mathcal{C}(\mathbb{X}(2,2,2,2,\lambda))$: cluster category of a Geigle-Lenzing weighted proj. line
Keller 2005, Barot-Kussin-Lenzing 2010, GKO 2013

Keller's talk

$\boxed{\text{sg}(R),\ R\text{: complete, local, isolated cDV sing. w/ small crepant res. Wemyss 2018}}$

...

$(d \geqslant 2)$ mod Π_{d+1}, Π_{d+1}: (d+1)-preproj. alg. of type $\vec{\mathbb{A}}_n$ Iyama-Oppermann 2013
mod A, A: self-injective d-Nakayama algebra J-Külshammer 2016
$\mathcal{C}(\Pi_{d+1}(A))$: d-CY cluster category of the derived (d+1)-preproj. alg. of a d-representation finite algebra A Iyama-Oppermann 2013