Deriving a theorem of Ladkani

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A theorem of Ladkani

R, S, E: DG rings (or, more generally, ring spectra)

$$ {}_S M_R \in \mathrm{D}\left(S^{\mathrm{op}} \otimes^L R\right) \text{ such that } M_R \in \operatorname{perf}(R) $$

$$ {}_E T_R \in \mathrm{D}\left(E^{\mathrm{op}} \otimes^L R\right) \text{ such that } - \otimes^L_E T \colon \mathrm{D}(E) \longrightarrow \mathrm{D}(R) \text{ is an equivalence} $$

Theorem (Ladkani 2011 for rings)

There is a derived equivalence

$$ \mathrm{D}\begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \simeq \mathrm{D}\begin{pmatrix} E & \mathrm{RHom}_R(M,T) \\ 0 & S \end{pmatrix} $$

Ladkani 2011: R, S, E rings, $M \in \mathrm{Mod}(S^{\mathrm{op}} \otimes^L R)$ and $\mathrm{RHom}_R(M,T) \in \mathrm{Mod}(E^{\mathrm{op}} \otimes S)$

Gluing along an exact functor

$F\colon \mathcal{B} \longrightarrow \mathcal{A}$ exact functor between stable ∞-categories

$$\begin{tikzcd}
\mathcal{L}_*(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "hPB"] & \mathrm{Fun}(s \to t, \mathcal{A}) \arrow[d, "s^*"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{L}^*(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "hPB"] & \mathrm{Fun}(s \to t, \mathcal{A}) \arrow[d, "t^*"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{A}
\end{tikzcd}$$

$\{\,(b, F(b) \xrightarrow{\varphi} a) \mid b \in \mathcal{B},\ \varphi$ in $\mathcal{A}\,\}$ $\qquad$ $\{\,(b, a \xrightarrow{\psi} F(b)) \mid b \in \mathcal{B},\ \psi$ in $\mathcal{A}\,\}$

The universal BGP reflection functors

Lemma (Folklore, proof below from Dyckerhoff-J-Walde 2019)

There are canonical, mutually inverse equivalences

$$S^- : \mathcal{L}^*(F) \overset{\simeq}{\longleftrightarrow} \mathcal{L}_*(F) : S^+$$

$$S^-(b, a \xrightarrow{\psi} F(b)) = (b, F(b) \to \operatorname{cone}(\psi)) \qquad S^+(b, F(b) \xrightarrow{\varphi} a) = (b, \operatorname{cocone}(\varphi) \to F(b))$$

$$\begin{tikzcd}
& \mathcal{B} \arrow[rr, "F"] \arrow[dd, equal] & & \mathcal{A} \arrow[dd, equal] \\
\mathcal{L}^*(F) \arrow[ur] \arrow[rr] \arrow[dd, dashed, shift left, "S^-"] & & \operatorname{Fun}(s \to t, \mathcal{A}) \arrow[ur, "t^*"'] \arrow[dd, shift left, "\operatorname{Cone}"] & \\
& \mathcal{B} \arrow[rr, "F"] & & \mathcal{A} \\
\mathcal{L}_*(F) \arrow[ur] \arrow[rr] \arrow[uu, dashed, shift left, "S^+"] & & \operatorname{Fun}(t \to u, \mathcal{A}) \arrow[ur, "t^*"'] \arrow[uu, shift left, "\operatorname{Cocone}"] &
\end{tikzcd}$$

Gluing along an adjunction

$F\colon \mathcal{B} \rightleftarrows \mathcal{A}\colon G$ adjunction $F \dashv G$ between stable ∞-categories

Lemma (Folklore)

There are canonical equivalences

$$\mathcal{L}_*(F) \xleftrightarrow{\simeq} \mathcal{L}^*(G)$$

$$(b, F(b) \xrightarrow{\varphi} a) \longmapsto (a, b \xrightarrow{\overline{\varphi}} G(a)) \qquad (a, b \xrightarrow{\psi} G(a)) \longmapsto (b, F(b) \xrightarrow{\overline{\psi}} a)$$

Remark. Identifying the corresp. hPB's with the ∞-category of sections of a bicartesian fibration classifying $F \dashv G$, we even have $\mathcal{L}_*(F) = \mathcal{L}^*(G)$

Proof of the theorem (J. 2019)

$${}_S M_R \in \mathrm{D}\left(S^{\mathrm{op}} \otimes^L R\right) \text{ s. t. } M_R \in \operatorname{perf}(R)$$

$${}_E T_R \in \mathrm{D}\left(E^{\mathrm{op}} \otimes^L R\right) \text{ such that}$$

$$- \otimes_E^L T \colon \ \mathrm{D}(E) \xrightarrow{\simeq} \mathrm{D}(R) \ \text{ is equiv.}$$

$$\begin{tikzcd}
\mathrm{D}(E) \arrow[r, "\mathrm{RHom}_R(M, - \otimes_E^L T)"] \arrow[d, "- \otimes_E^L T"'] & \mathrm{D}(S) \arrow[d, equal] \\
\mathrm{D}(R) \arrow[r, "\mathrm{RHom}_R(M, -)"] & \mathrm{D}(S)
\end{tikzcd}$$

$$\mathrm{RHom}_R(M, - \otimes_E^L T) \simeq - \otimes_E^L \mathrm{RHom}_R(M, T)$$

since $M_R \in \operatorname{perf}(R)$ (Eilenberg–Watts)

$$\begin{aligned}
\mathrm{D}\begin{pmatrix} S & M \\ 0 & R \end{pmatrix} & \overset{(1)}{\simeq} \mathcal{L}_*(- \otimes_S^L M) \\
\text{adjunction} \quad & \simeq \mathcal{L}^*(\mathrm{RHom}_R(M, -)) \\
\text{BGP reflection} \quad & \simeq \mathcal{L}_*(\mathrm{RHom}_R(M, -)) \\
\text{functoriality} \quad & \simeq \mathcal{L}_*(- \otimes_E^L \mathrm{RHom}_R(M, T)) \\
& \overset{(2)}{\simeq} \mathrm{D}\begin{pmatrix} E & \mathrm{RHom}_R(M, T) \\ 0 & S \end{pmatrix}
\end{aligned}$$

Q.E.D.

(1), (2):
Recognition Theorem
- Keller 1994 (DG rings)
- Schwede–Shipley 2003 (ring spectra)

Simple computation using assoc. recollement

Extra: Recollement

$$\mathrm{D}\begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \simeq \mathcal{L}_*(- \otimes^{L}_{S} M)$$

$$\begin{tikzcd}
\mathrm{D}(R) = \mathcal{A} \arrow[r, hook, "i" description] & \mathcal{L}_*(F) \arrow[l, two heads, bend right, "i_L"'] \arrow[l, two heads, bend left, "i_R"] \arrow[r, two heads, "p" description] & \mathcal{B} = \mathrm{D}(S) \arrow[l, hook, bend right, "p_L"'] \arrow[l, hook, bend left, "p_R"]
\end{tikzcd}$$

compact gen: $i(R) \oplus p_L(S)$

$$\mathrm{im}(i) = \ker(p)$$

$$i_L \dashv i \dashv i_R \qquad p_L \dashv p \dashv p_R$$

i, p_L, p_R are fully faithful

$$i_L(b, F(b) \xrightarrow{\varphi} a) \simeq \mathrm{cone}\,(\varphi)$$

$$p_L(b) = (b, F(b) \xrightarrow{1} F(b))$$

$$i(a) = (0, F(0) \to a)$$

$$p(b, F(b) \xrightarrow{\varphi} a) = b$$

$$i_R(b, F(b) \xrightarrow{\varphi} a) = a$$

$$p_R(b) = (b, F(b) \to 0)$$

$$(i_R \circ p_L)(b) = F(b) \quad \text{and} \quad (i_L \circ p_R)(b) \simeq F(b)[1]$$

F has right adjoint $\iff$ i_R has right adjoint

Thank you for your attention!

Further reading:

Dyckerhoff, J., Walde – Generalised BGP reflection functors via the Grothendieck construction.
10.1093/imrn/rnz194

Ladkani – Derived equivalences of triangular matrix rings arising from extensions of tilting modules.
10.1007/s10468-009-9175-0