# Homotopical algebra in exact (∞-)cat's

(jt. work in progress with Kvamme, Palu & Walde)

## § Motivation

$(\mathcal{E}, \mathcal{S})$ : Frobenius exact category

additive category — class of "admissible" short. ex. seq.

e.g. $R$ = quasi-Frobenius ring

$\mathcal{E} = \mathrm{Mod}_R$ , $\mathcal{S}$ = all. s.e.s

Δ'd cat's

$\underline{\mathcal{E}}_{\mathcal{S}} := \mathcal{E}/[\mathcal{P}]$ : stable cat , $\mathcal{P}$ = $\mathcal{S}$-projectives

$D^b(\mathcal{E}, \mathcal{S}) := K^b(\mathcal{E}) / \mathcal{S}$-acyclic complexes

**Thm** (Buchweitz 1986, Keller–Vossieck 1988) $\mathcal{E}$ weakly idempotent-comp

∃ can. exact equiv. $\underline{\mathcal{E}}_{\mathcal{S}} \xrightarrow{\simeq} D^b(\mathcal{E}, \mathcal{S}) / \mathrm{thick}\, \mathcal{P}$

**Today** Variants of Buchweitz's theorem

\+ refined universal properties of loc. functors

→ objects with better properties!

**Problem** 1-cat localisations are often ill-behaved

## § $\infty$-categories & their localisations

$\mathcal{C}$: $\infty$-category

$X, Y \in \mathcal{C} \rightsquigarrow \mathrm{Map}_{\mathcal{C}}(X,Y)$: "space" of morphisms

$\mathrm{Ho}(\mathcal{C})$: homotopy category of $\mathcal{C}$

$\mathrm{Ho}(\mathcal{C})(X,Y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(X,Y)$: set of path conn. components

**Def / Prop** $W$: class of morphisms in $\mathcal{C}$

$\Longrightarrow \exists\ \mathcal{C}[W^{-1}]$ <u>$\infty$-cat localisation</u> of $\mathcal{C}$ at $W$

& $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$ s.t. $\forall \mathcal{D}$ = $\infty$-cat

$$\gamma^* : \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_W(\mathcal{C}, \mathcal{D})$$

$\mathrm{Fun}_W(\mathcal{C}, \mathcal{D})$ : $F : \mathcal{C} \to \mathcal{D}$ st. $\forall f \in W$ $F(f)$: invertible

**Prop** $\exists$ can $\mathrm{Ho}(\mathcal{C})[W^{-1}] \xrightarrow{\simeq} \mathrm{Ho}(\mathcal{C}[W^{-1}])$

**Example** $0 \neq R$: ring $\mathcal{C} = C(\mathrm{Mod}_R)$, $W$ = q-iso's

$\mathcal{C}[W^{-1}] =: \mathcal{D}(\mathrm{Mod}_R)$: derived $\infty$-cat of $R$ — not equiv.!

$\mathrm{Ho}(\mathcal{C}[W^{-1}]) =: D(\mathrm{Mod}_R)$: derived 1-cat of $R$

# § Exact $\infty$-categories

**Def** (Lurie 2006) $\mathcal{C}$ : $\infty$-cat is stable if

(1) $\exists\, 0 \in \mathcal{C}$ : zero object

(2) $\forall f : X \to Y$ in $\mathcal{C}$ $\exists$

htpy. PO  &  htpy. PB

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & 0 \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd} \quad \& \quad \begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

(3) A square in $\mathcal{C}$ of the form

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

is homotopy pushout

$\Updownarrow$

is homotopy pullback

**Thm** (Lurie) $\mathcal{C}$ : stable $\infty$-cat

canonical

$\Longrightarrow$ $\mathrm{Ho}(\mathcal{C})$ has the structure of a triang. cat

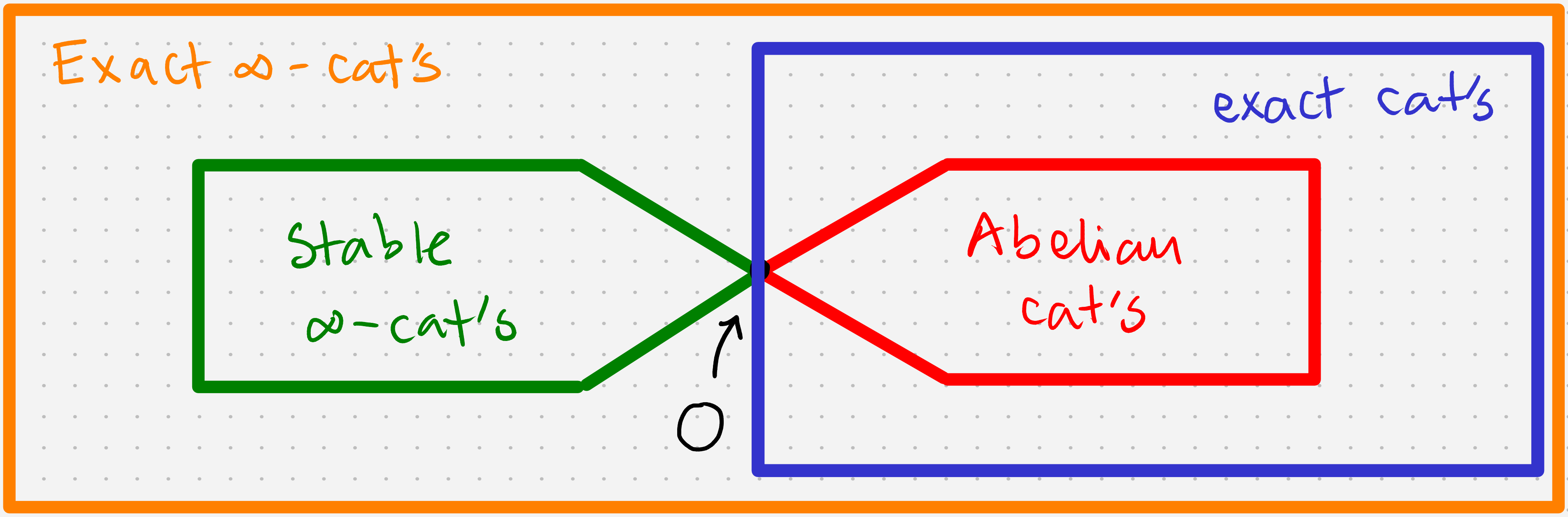
**Examples** $\mathcal{C}$ : stable $\infty$-cat

(1) $(\mathcal{E}, \mathcal{S})$ : exact cat & $\mathcal{C} = \mathcal{D}(\mathcal{E}, \mathcal{S})$ : derived $\infty$-cat

e.g. $\mathcal{E} = \mathrm{Mod}_R$, $\mathcal{S} = \mathcal{S}_{max}$ & $\mathcal{C} = \mathcal{D}(\mathrm{Mod}_R)$

(2) $(\mathcal{E}, \mathcal{S})$ : Frobenius exact cat & $\mathcal{C} = \underline{\mathcal{E}}_{\mathcal{S}} := \mathcal{E}[\mathrm{st.eq}^{-1}]$

e.g. $R$ : QF ring, $\mathcal{E} = \mathrm{Mod}_R$, $\mathcal{S} = \mathcal{S}_{max}$, $\mathcal{C} = \underline{\mathrm{Mod}}_R$




**Def** (Lurie) $\mathcal{A}$ : $\infty$-cat is <u>additive</u> if

(1) $\exists\, 0 \in \mathcal{A}$ : zero object $\left( \forall X \in \mathcal{A} \;\; \mathrm{Map}(X,0) \simeq * \simeq \mathrm{Map}(0,X) \right)$

(2) $\mathcal{A}$ admits finite $\amalg$'s & $\Pi$'s

(3) $\forall X, Y \in \mathcal{A} \quad X \amalg Y \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} X \Pi Y$ is an equiv.

(4) $\forall X \in \mathcal{A} \quad X \oplus X \xrightarrow{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} X \oplus X$ is an equiv.

**Rmk** $\mathcal{A}$ : additive $\infty$-cat $\Rightarrow$ $\mathrm{Ho}(\mathcal{A})$ : additive cat

**Examples** $\mathcal{A}$ : additive $\infty$-cat where

(1) $\mathcal{A}$ : additive 1-category (e.g. abelian cat's)

(2) $\mathcal{A}$ : stable $\infty$-category

(3) $\mathcal{B}$ : additive $\infty$-cat & $\mathcal{A} \subseteq \mathcal{B}$ closed under fin $\oplus$'s

**Def** (Barwick 2015)

admissible seq.

An **exact $\infty$-cat** is a pair $(\mathcal{E}, \mathcal{S})$ where

(1) $\mathcal{E}$: additive $\infty$-cat

$p \circ i \underset{h}{\simeq} 0$

(2) $$\mathcal{S} = \left\{ \begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, two heads, "p"] \\ 0 \arrow[r] & Z \end{tikzcd} \right\}$$ is an exact structure on $\mathcal{E}$

$h$: PO + PB

**Examples** $(\mathcal{E}, \mathcal{S})$: exact $\infty$-cat where

(1) $(\mathcal{E}, \mathcal{S})$: exact 1-cat

(2) $\mathcal{E}$: stable $\infty$-cat, $\mathcal{S} = \mathcal{S}_{max}$: all $\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$

**Def** $(\mathcal{E}, \mathcal{S}_{\mathcal{E}})$ & $(\mathcal{F}, \mathcal{S}_{\mathcal{F}})$: exact $\infty$-cat's

$F: \mathcal{E} \to \mathcal{F}$ is <u>exact</u> if $F$: additive & $F(\mathcal{S}_{\mathcal{E}}) \subseteq \mathcal{S}_{\mathcal{F}}$

**Thm** (Klemenc 2020) $(\mathcal{E}, \mathcal{S})$: exact $\infty$-cat

$\Rightarrow \exists\ \iota : \mathcal{E} \xhookrightarrow[\text{exact}]{\text{f.f}} \mathcal{H}^{st}(\mathcal{E}, \mathcal{S})$: stable s.t. $\forall \mathcal{C}$: stable

$$\iota^* : \mathrm{Fun}^{ex}(\mathcal{H}^{st}(\mathcal{E}, \mathcal{S}), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{ex}(\mathcal{E}, \mathcal{C})$$

**Thm** (Bunke-Cisinski-Kasprowski-Winges 2019)

$(\mathcal{E}, \mathcal{S})$ : exact 1-cat $\Rightarrow$ $\exists$ can $\mathcal{H}^{st}(\mathcal{E},\mathcal{S}) \xrightarrow{\sim} \mathcal{D}^{b}(\mathcal{E},\mathcal{S})$

$\mathcal{E}$ $\mathcal{S}$

**Thm** (J-Kvamme-Palu-Walde)

$(\mathcal{E}, \mathcal{S})$ : exact $\infty$-cat $\Rightarrow$ $\exists$ can $\mathcal{H}^{st}(\mathcal{E},\mathcal{S}) \xrightarrow{\sim} \mathcal{D}^{b}(\mathcal{E},\mathcal{S})$

where $\mathcal{D}^{b}(\mathcal{E},\mathcal{S})$ is an appropriate stable $\infty$-cat constructed in terms of coherent (!) complexes

$(\mathcal{E}, \mathcal{S})$ : exact $\infty$-cat. The following notions from the theory of 1-cat's :

* $\mathcal{S}$-projective, $\mathcal{S}$-injective, Frobenius.
* weakly idempotent-complete

**Prop** (JKPW, cf. Nakaoka-Palu 2019)

$(\mathcal{E}, \mathcal{S})$ : Frobenius exact $\infty$-cat. TFAE

(1) $\mathcal{E}$ : stable $\infty$-cat & $\mathcal{S} = \mathcal{S}_{max}$

(2) $P \in \mathcal{E}$ is $\mathcal{S}$-projective $\Leftrightarrow$ $P \simeq 0$

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

## § Resolving subcategories   $(\mathcal{E}, \mathcal{S})$ : weakly idemp. cpt.

**Def** (Auslander-Bridger 1969)

$\mathcal{A} \subseteq \mathcal{E}$ is <u>weakly</u> <u>resolving</u> if

(1) $\mathcal{A}$ is closed under extensions

(2) $\forall X \in \mathcal{E} \;\; \exists\, Y \rightarrowtail A \twoheadrightarrow X$, $A \in \mathcal{A}$

(3) $X \rightarrowtail A' \twoheadrightarrow A$, $A, A' \in \mathcal{A} \implies X \in \mathcal{A}$

deflations

inflations

$\mathcal{A}$ is <u>resolving</u> if in addition

(4) $\mathcal{A}$ is closed under direct summands

**Examples** $R$: ring

(1) $\mathcal{E} = \mathrm{Mod}_R$, $\mathcal{S} = \mathcal{S}_{max}$, $\mathcal{A} = \mathrm{Proj}_R$

(2) $\mathcal{E} = \mathrm{Mod}_R$, $\mathcal{S} = \mathcal{S}_{max}$, $\mathcal{A} = \mathrm{GProj}_R$

(3) $\mathcal{E} = C(\mathrm{Mod}_R)$, $\mathcal{S} = \mathcal{S}_{max}$, $\mathcal{A}$ = acyclic cpx's

**Notation** $\mathcal{A} \subseteq \mathcal{E} \rightsquigarrow W_{\mathcal{A}} = f : X \to Y$ in $\mathcal{E}$

such that $\exists\, X \overset{\binom{f}{*}}{\rightarrowtail} Y \oplus A' \twoheadrightarrow A$, $A, A' \in \mathcal{A}$

**Def** (JKPW, Cisinski) $W$: class of morphisms in $\mathcal{E}$

$(\mathcal{E}, \mathcal{S}, W)$ is an exact ($\infty$-)cat of fibrant obj if

(1) $W$ contains the identities

is closed under composition

and satisfies the 2-out-3 property

$$\left( \begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, bend left=20, "="] & & Z \end{tikzcd} \quad 2/3 \text{ of } f, g, h \text{ in } W \Rightarrow 3/3 \text{ in } W \right)$$

(2) $\forall f : X \to Y$ in $\mathcal{E}$, $\exists$
$$\begin{tikzcd} & X' \arrow[dr, two heads] & \\ X \arrow[ur, "w"] \arrow[rr, "f"'] \arrow[rr, phantom, bend left=20, "="] & & Y \end{tikzcd} \quad w \in W$$

htpy pullback

(3)
$$\begin{tikzcd} Y' \arrow[r] \arrow[d, two heads, "W \ni !"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, two heads, "\forall w \in W \cap \twoheadrightarrow"] \\ Z' \arrow[r, "\forall"'] & Z \end{tikzcd}$$

**Notation** $W$: class of morphisms in $\mathcal{E}$

$$\mathcal{A}_W := \{ A \in \mathcal{E} \mid A \twoheadrightarrow 0 \text{ lies in } W \}$$

**Thm** (JKPW) $(\mathcal{E}, \mathcal{S})$: weakly idemp. cpt exact $\infty$-cat.

The associations

$$\mathcal{A} \longmapsto W_{\mathcal{A}} \quad \& \quad W \longmapsto \mathcal{A}_W$$

yield mutually inverse bijections between:

(1) weakly resolving subcategories $\mathcal{A} \subseteq \mathcal{E}$

(2) classes of morphisms $W$ in $\mathcal{E}$ such that $(\mathcal{E}, \mathcal{S}, W)$ is an exact ($\infty$-)category of fibrant objects satisfying some (mild) technical conditions

Moreover, TFAE

* $\mathcal{A} \subseteq \mathcal{E}$ is resolving
* $W_{\mathcal{A}} = \overline{W_{\mathcal{A}}} := \{ f \text{ in } \mathcal{E} \mid f: \text{invertible in } \mathcal{E}[W_{\mathcal{A}}^{-1}] \}$

**Aim** $\mathcal{A} \subseteq \mathcal{E}$: weakly resolving

Study $\infty$-cat localisation $\mathcal{E}[W_{\mathcal{A}}^{-1}]$

**Thm** (JKPW + Cisinski / Lurie)

$\mathcal{A} \subseteq \mathcal{E}$ : weakly resolving TFSH

(1) $\mathcal{E}[W_{\mathcal{A}}^{-1}]$ is additive and admits pullbacks

(2) $\mathrm{Ho}(\mathcal{E}[W_{\mathcal{A}}^{-1}])$ has the structure of a left triangulated category in the sense of Keller & Vossieck.

**Thm** (JKPW + BCKW / Klemenc + Cisinski)

$\mathcal{A} \subseteq \mathcal{E}$ : weakly resolving TFSH

(1) $\exists$ can. equivalence

$$\mathrm{SW}(\mathcal{E}[W_{\mathcal{A}}^{-1}]) \xrightarrow{\simeq} \mathcal{D}^b(\mathcal{E}, \mathcal{S}) / \mathrm{thick}\,\mathcal{A}$$

stabilisation

(2) $\forall\, \mathcal{C}$ : stable $\infty$-category there is a canonical equivalence

$$\mathrm{Fun}^{ex}(\mathrm{SW}(\mathcal{E}[W_{\mathcal{A}}^{-1}]), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{ex}_{\mathcal{A}}(\mathcal{E}, \mathcal{C})$$

$\mathrm{Fun}^{ex}_{\mathcal{A}}(\mathcal{E}, \mathcal{C}) = F : \mathcal{E} \to \mathcal{C}$ exact st. $F|_{\mathcal{A}} = 0$

**Def** $\mathcal{A} \subseteq \mathcal{E}$ is <u>weakly biresolving</u> if both $\mathcal{A} \subseteq \mathcal{E}$ & $\mathcal{A}^{op} \subseteq \mathcal{E}^{op}$ are weakly resolving. We call $\mathcal{A}$ <u>biresolving</u> if $\mathcal{A}$ is closed under direct summands.

**Def** $W$: class of morphisms in $\mathcal{E}$
$(\mathcal{E}, \mathcal{S}, W)$ is an <u>exact $\infty$-cat of bifibrant obj.</u>
if $(\mathcal{E}, \mathcal{S}, W)$ & $(\mathcal{E}^{op}, \mathcal{S}^{op}, W^{op})$ are exact $\infty$-categories of fibrant objects.

**Thm** (JKPW)
There is a bijective correspondence between:
(1) weakly biresolving subcategories $\mathcal{A} \subseteq \mathcal{E}$
(2) classes of morphisms $W$ in $\mathcal{E}$ such that $(\mathcal{E}, \mathcal{S}, W)$ is an exact ($\infty$-)category of bifibrant objects (without any further assumptions)

**Examples** $R$ : ring

(1) $R$ : Frobenius, $\mathcal{E} = \mathrm{Mod}_R$, $\mathcal{S} = \mathcal{S}_{max}$, $\mathcal{A} = \mathrm{Proj}_R$

(2) $\mathcal{E} = C(\mathrm{Mod}_R)$, $\mathcal{S} = \mathcal{S}_{max}$, $\mathcal{A}$ = acyclics

(3) $(\mathcal{E}, \mathcal{S})$ : Frobenius ex. $(\infty,1)$ cat, $\mathcal{A} = \mathcal{S}$-Proj

**Thm** (JKPW + Cisinski/Lurie)

$\mathcal{A} \subseteq \mathcal{E}$ : weakly biresolving TFSH

(1) $\mathcal{E}[W_{\mathcal{A}}^{-1}]$ is a stable $\infty$-cat.

(2) $\mathrm{Ho}(\mathcal{E}[W_{\mathcal{A}}^{-1}])$ has the structure of a triangulated category. (Rump 2021 for exact 1-cat's)

**Thm** (JKPW + BCKW/Klemenc)

$\mathcal{A} \subseteq \mathcal{E}$ : weakly biresolving TFSH

(1) $\exists$ can eq. $\mathcal{E}[W_{\mathcal{A}}^{-1}] \xrightarrow{\simeq} \mathcal{D}^b(\mathcal{E}, \mathcal{S}) / \mathrm{thick}\, \mathcal{A}$

(2) $\forall\, \mathcal{C}$ : stable $\infty$-cat. there is a can equiv

$$\mathrm{Fun}^{ex}(\mathcal{E}[W_{\mathcal{A}}^{-1}], \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{ex}_{\mathcal{A}}(\mathcal{E}, \mathcal{C})$$

## § Complete relative resolving pairs  $(\mathcal{E}, \mathcal{S})$: weakly idemp. cpt.

**Def** (JKPW) A complete relative resolving pair is a pair $(\mathcal{A}, \mathcal{F})$ of ext-closed subcat's of $\mathcal{E}$ st.

(1) $\forall F \in \mathcal{F} \ \exists \ X \rightarrowtail A \twoheadrightarrow F, \ A \in \mathcal{A}$

(2) $\forall F \rightarrowtail A' \twoheadrightarrow A, \ A, A' \in \mathcal{A} \ \& \ F \in \mathcal{F} \Rightarrow F \in \mathcal{A}$

(3) $\forall X \in \mathcal{A} \ \exists \ X \rightarrowtail F \twoheadrightarrow A, \ A \in \mathcal{A} \ \& \ F \in \mathcal{F}$

**Examples** $(\mathcal{A}, \mathcal{F})$ CCRP where

(1) $\mathcal{A} \subseteq \mathcal{E}$: weakly resolving & $\mathcal{F} = \mathcal{E}$.

(2) $\mathcal{X} \subseteq \mathcal{E}$: weakly coresolving

$\mathcal{A}$: $\mathcal{S}$-acyclic complexes in $C^+(\mathcal{E})$

$\mathcal{F}$: $C^+(\mathcal{X}) \subseteq C^+(\mathcal{E})$

$(\mathcal{A}, \mathcal{F})$ is a CCRP in $C^+(\mathcal{E})$

(3) $R$: ring, $\mathcal{E} = C(\mathrm{Mod}_R)$, $\mathcal{S} = \mathcal{S}_{\max}$

$\mathcal{A}$: acyclic complexes

$\mathcal{F}$: DG-Injective complexes $I$ $\left\{ \begin{tikzcd} L \arrow[r, tail, "\forall", "\mathrm{qiso}"'] \arrow[d, "\forall"'] \arrow[dr, phantom, "="] & M \arrow[dl, dashed, "\exists"] \\ I & \end{tikzcd} \right.$

**Notation** $(\mathcal{A}, \mathcal{F})$ subcategories of $\mathcal{E}$

$\mathrm{Fib}_{(\mathcal{A},\mathcal{F})} := \{ Y \twoheadrightarrow Z \mid F \rightarrowtail Y \twoheadrightarrow Z,\ F \in \mathcal{F} \}$

$W_{(\mathcal{A},\mathcal{F})}$ : smallest class of morphisms

+ closed under comp. & containing identities
+ satisfying 2-out-of-3
+ containing $\{ X \rightarrowtail Y \mid X \rightarrowtail Y \twoheadrightarrow A,\ A \in \mathcal{A} \}$

**Thm** (JKPW) $(\mathcal{E}, \mathcal{S})$: weakly idemp. cpt. exact $\infty$ cat

The association $(\mathcal{A}, \mathcal{F}) \longmapsto (W_{(\mathcal{A},\mathcal{F})}, \mathrm{Fib}_{(\mathcal{A},\mathcal{F})})$

yields a bijection between

(1) CRRP's $(\mathcal{A}, \mathcal{F})$ in $(\mathcal{E}, \mathcal{S})$

(2) Pairs $(W, \mathrm{Fib})$ such that $(\mathcal{E}, \mathcal{S}, W, \mathrm{Fib})$

is an exact fibration $\infty$-category

satisfying (mild) technical conditions

**Prop** (JKPW) $(\mathcal{A}, \mathcal{F})$ CRRP in $\mathcal{E}$

$\Rightarrow$ $\mathcal{A} \cap \mathcal{F} \subseteq \mathcal{F}$ is weakly resolving

**Coro** (JKPW + Cisinski) $(\mathcal{A}, \mathcal{F})$ CRRP in $\mathcal{E}$

$\Rightarrow$ $\exists$ can eq. $\mathcal{F}[W_{\mathcal{A}\cap\mathcal{F}}^{-1}] \xrightarrow{\simeq} \mathcal{E}[W_{(\mathcal{A},\mathcal{F})}^{-1}]$

**Example**

(1) $R$: ring, $\mathcal{E} = C(\mathrm{Mod}_R)$, $\mathcal{S} = \mathcal{S}_{max}$

$\mathcal{A}$ = acyclic complexes

$\mathcal{F}$ : DG-Injective complexes

$W_{(\mathcal{A},\mathcal{F})}$ : quasi-isomorphisms

$W_{\mathcal{A}\cap\mathcal{F}}$ : homotopy equivalences

$$\mathrm{DG\text{-}Inj}(R)[\mathrm{heq}^{-1}] \xrightarrow{\simeq} C(\mathrm{Mod}_R)[\mathrm{qiso}^{-1}]$$

(2) $\mathcal{X} \subseteq \mathcal{E}$ weakly coresolving

$\mathcal{A}$ : $\mathcal{S}$-acyclic complexes in $C^+(\mathcal{E})$

$\mathcal{F}$ : $C^+(\mathcal{X}) \subseteq C^+(\mathcal{E})$

$$\mathcal{D}^+(\mathcal{X}, \mathcal{S}|_{\mathcal{X}}) \xrightarrow{\simeq} \mathcal{D}^+(\mathcal{E}, \mathcal{S})$$