Generalised BGP reflection functors
(jt. with Tobias Dyckerhoff & Tashi Walde)

§ Motivation

THM (Happel 1987, Auslander-Platzeck-Reiten 1979, Bernstein-Gelfand-Ponomarev 1973)

Q : finite quiver & $s \in Q_0$: source
Q' : quiver obtained from Q by reversing all arrows incident to s

$$\Longrightarrow \quad D(\mathrm{Mod}(kQ)) \underset{\Delta}{\simeq} D(\mathrm{Mod}(kQ')) , \quad k: \text{comm. ring}$$

THM (Ladkani 2008)

P : finite poset with maximal element ∞
P': poset obtained from P by removing ∞ and adding a minimal element $-\infty$.

$$\Longrightarrow \quad D(\mathrm{Fun}(P, \mathcal{A})) \underset{\Delta}{\simeq} D(\mathrm{Fun}(P', \mathcal{A})) , \quad \mathcal{A}: \text{abelian cat.}$$

THM (Rahn - Stovicek 2018)

Q, Q' as in Happel's theorem (can replace Q by small cat.)

$$\Longrightarrow \quad \mathrm{Fun}(Q, \mathcal{D}) \simeq \mathrm{Fun}(Q', \mathcal{D}) , \quad \mathcal{D}: \text{enhanced tri. cat.}$$

TODAY Unified construction of (generalisations) of these equivalences

§ Stable ∞-categories (enhanced tri. categories)

$\mathcal{D}$: stable ∞-category

$\rightsquigarrow$ $\forall X, Y \in \mathcal{D}$, $\mathrm{Map}_{\mathcal{D}}(X, Y)$: "space" of maps $X \to Y$

$\rightsquigarrow$ $\mathrm{Ho}(\mathcal{D})$: homotopy category
$\mathrm{Ho}(\mathcal{D})(X, Y) := \pi_0 \mathrm{Map}_{\mathcal{D}}(X, Y)$: set of path connected comp.

THM (Lurie 2006) $\mathcal{D}$: stable ∞-cat. TFSH

(1) $\mathrm{Ho}(\mathcal{D})$ is (canonically) a tri. cat.

(2) $\forall A$: small cat, $\mathrm{Fun}(A, \mathcal{D})$ is a stable ∞-cat.

(3) Have mutually inverse equivalences
$\mathrm{Cone} : \mathrm{Fun}(0 \to 1, \mathcal{D}) \rightleftarrows \mathrm{Fun}(0 \to 1, \mathcal{D}) : \mathrm{Cocone}$

PROP (Lurie) A : small cat & $\mathcal{G}$: Grothendieck category

$\Longrightarrow$ $\mathrm{Fun}(A, \mathcal{D}(\mathcal{G})) \simeq \mathcal{D}(\mathrm{Fun}(A, \mathcal{G}))$
↑ derived ∞-category

RMK All tri. cat's that arise in conventional mathematical practice (e.g. derived cat's, stable module cat's, ...) arise as the homotopy cat. of some (not necessarily unique) stable ∞-cat.

RMK Pre-triangulated DG k-cat's $\simeq$ k-linear stable ∞-cat's

§ Upper-triangular glueing

$F : \mathcal{D} \longrightarrow \mathcal{C}$ exact functor between stable ∞-cat's.

$$\begin{tikzcd}
\mathcal{L}_*(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "hPB"] & \mathrm{Fun}(s \to t, \mathcal{C}) \arrow[d, "s^*"] \\
\mathcal{D} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd}$$

$$\mathcal{L}_*(F) = \{ (d, Fd \xrightarrow{\varphi} c) \mid d \in \mathcal{D} \ \& \ \varphi \text{ in } \mathcal{C} \}$$

EXAMPLE R, S : rings & ${}_S M_R$: bimodule

$- \overset{\mathbb{L}}{\underset{S}{\otimes}} M : \mathcal{D}(\mathrm{Mod}\, S) \to \mathcal{D}(\mathrm{Mod}\, R)$

$$\Longrightarrow \mathcal{L}_*(- \overset{\mathbb{L}}{\underset{S}{\otimes}} M) \simeq \mathcal{D}\left(\mathrm{Mod} \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}\right)$$ (hence the name)

—— DUAL CONSTRUCTION ——

$$\begin{tikzcd}
\mathcal{L}^*(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "hPB"] & \mathrm{Fun}(s \to t, \mathcal{C}) \arrow[d, "t^*"] \\
\mathcal{D} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd}$$

$$\mathcal{L}^*(F) = \{ (d, c \xrightarrow{\psi} Fd) \mid d \in \mathcal{D} \ \& \ \psi \text{ in } \mathcal{C} \}$$

LEMMA (Folklore) There are mutually inverse equivalences

$$S^- : \mathcal{L}^*(F) \rightleftarrows \mathcal{L}_*(F) : S^+$$

$$S^-(d, c \xrightarrow{\psi} Fd) = (d, Fd \to \mathrm{cone}\, \psi), \quad S^+(d, Fd \xrightarrow{\varphi} c) = (d, \mathrm{cocone}\, \varphi \to Fd)$$

PROOF (DJW 2019)

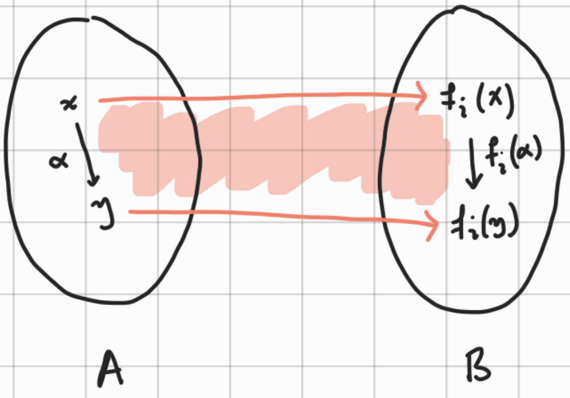
$$\begin{tikzcd}
 & \mathcal{D} \arrow[rr, "F"] \arrow[dd, equal] & & \mathcal{C} \arrow[dd, equal] \\
\mathcal{L}^*(F) \arrow[ur] \arrow[rr] \arrow[dd, dashed, shift right, "\bar{s}"'] & & \mathrm{Fun}(s \to t, \mathcal{C}) \arrow[ur, "t^*"'] \arrow[dd, shift right, "\mathrm{Cone}"'] & \\
 & \mathcal{D} \arrow[rr, "F"] & & \mathcal{C} \\
\mathcal{L}_*(F) \arrow[ur] \arrow[rr] \arrow[uu, dashed, shift right, "s^+"'] & & \mathrm{Fun}(t \to u, \mathcal{C}) \arrow[ur, "t^*"'] \arrow[uu, shift right, "\mathrm{cocone}"'] &
\end{tikzcd}$$

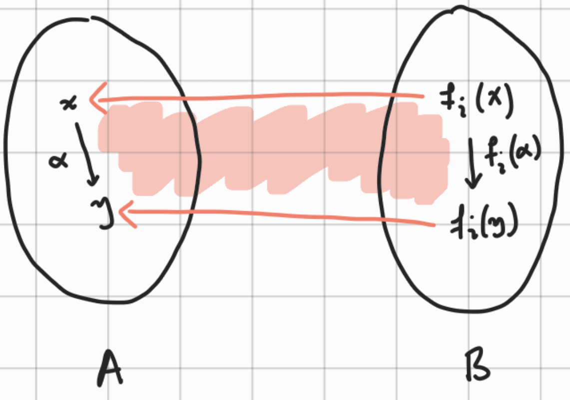
▨

§ Generalised BGP reflection functors

A, B : small cat's, $\underline{f} = (f_1, \dots, f_n) : \coprod_{i=1}^{n} A \longrightarrow B$ functor

$$\begin{tikzcd}
\int_n \underline{f} & A \times \left\{ 0 \to 1, \dots, 0 \to n \right\} \arrow[l] \\
B \arrow[u] \arrow[ur, phantom, "\mathrm{PO}"] & \coprod_{i=1}^{n} A \arrow[l, "\underline{f}"] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\int^n \underline{f} & A \times \left\{ 0 \leftarrow 1, \dots, 0 \leftarrow n \right\} \arrow[l] \\
B \arrow[u] \arrow[ur, phantom, "\mathrm{PO}"] & \coprod_{i=1}^{n} A \arrow[l, "\underline{f}"] \arrow[u]
\end{tikzcd}$$



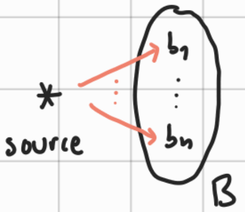


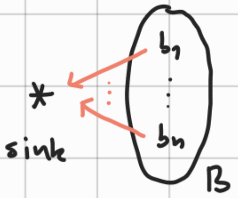
RMK $\int_n \underline{f}$ & $\int^n \underline{f}$ are the two variants of the Grothendieck construction of the diagram $A \underset{f_n}{\overset{f_1}{\substack{\longrightarrow \\ \vdots \\ \longrightarrow}}} B$

EXAMPLES

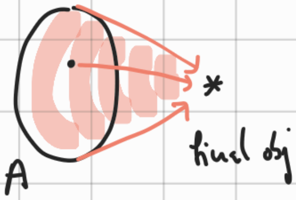
(1) $A = \{*\}$, $b_1, \dots, b_n \in B$

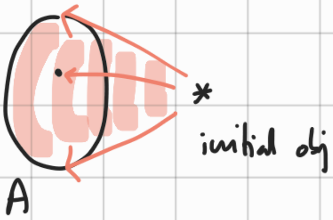
$\underline{b} = (b_1, \dots, b_n) : \coprod_{i=1}^{n} \{*\} \to B$





(2) $B = \{*\}$, $f : A \longrightarrow \{*\}$





THM (DJW 2019) $\mathcal{D}$: stable ∞-cat

A , B : small cat's , $\underline{f} = (f_1, \dots, f_n) : \coprod_{i=1}^{n} A \longrightarrow B$ functor

$$F : \mathrm{Fun}(B, \mathcal{D}) \longrightarrow \mathrm{Fun}(A, \mathcal{D})$$

$$M \longmapsto \left(a \longmapsto \bigoplus_{i=1}^{n} M_{f_i(a)} \right)$$

$\Longrightarrow$
$$\begin{tikzcd} \mathrm{Fun}(\mathbb{S}_n \underline{f}, \mathcal{D}) \arrow[r, phantom, "\overset{!}{\simeq}"] & \mathcal{L}^*(F) \arrow[d, shift right, "s^-"'] \\ \mathrm{Fun}(\mathbb{S}^n \underline{f}, \mathcal{D}) \arrow[r, phantom, "\overset{!}{\simeq}"] & \mathcal{L}_*(F) \arrow[u, shift right, "s^+"'] \end{tikzcd}$$

from Lemma

RMK The proof of the theorem is rather simple and relies only on basic (higher-) categorical principles.

RMK Suitable choices of A, B & $\underline{f}$ reproduce the results of Ladkani & Rahn-Stoviceck (see next page).

RMK In the terminology of Rahn-Stovicek, the above thm is a statement in "abstract rep. theory", that is rep. theory of quivers & cat's in arbitrary enhanced tri. cat's.

EXAMPLES

(1) $A = \{*\}$, $\underline{b} = (b_1, \dots, b_n) : \coprod_{i=1}^{n} \{*\} \longrightarrow B$

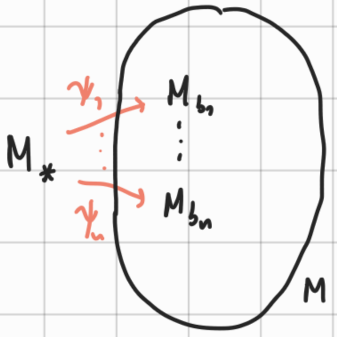
$\mathrm{Fun}(A, \mathcal{D}) = \mathrm{Fun}(\{*\}, \mathcal{D}) \simeq \mathcal{D}$

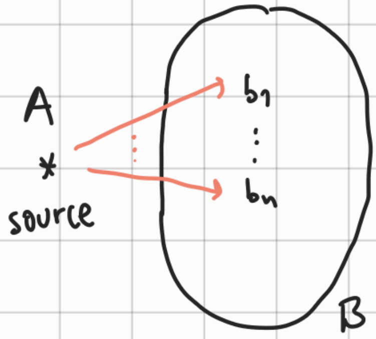
$$\begin{aligned} F : \mathrm{Fun}(B, \mathcal{D}) &\longrightarrow \mathcal{D} \\ M &\longmapsto \bigoplus_{i=1}^{n} M_{b_i} \end{aligned}$$

$\mathcal{L}^*(F) = \{ (M, M_* \xrightarrow{\psi} \bigoplus_{i=1}^{n} M_{b_i}) \mid M \in \mathrm{Fun}(B, \mathcal{D}) \ \& \ M_* \in \mathcal{D} \}$

$\updownarrow \wr$

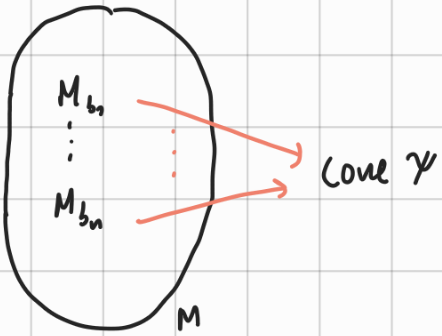
$\mathrm{Fun}(\int_n \underline{b}, \mathcal{D}) \ni$

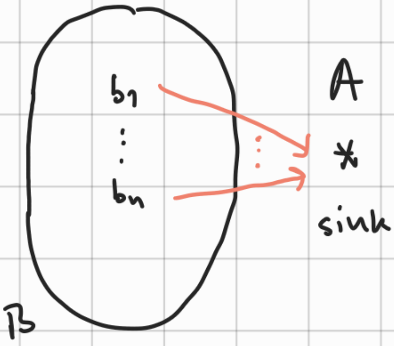




$\downarrow \bar{S}$ $\qquad$ $\mapsto$

$\mathrm{Fun}(\int^n \underline{b}, \mathcal{D}) \ni$





$\updownarrow \wr$

$\mathcal{L}_*(F) = \{ (M, \bigoplus_{i=1}^{n} M_{b_i} \to M_*) \mid M \in \mathrm{Fun}(B, \mathcal{D}), M_* \in \mathcal{D} \}$

This recovers the BGP reflection functors of Rahn-Stovicek

(2) $\mathcal{B} = \{*\}$, $f : A \longrightarrow \{*\}$

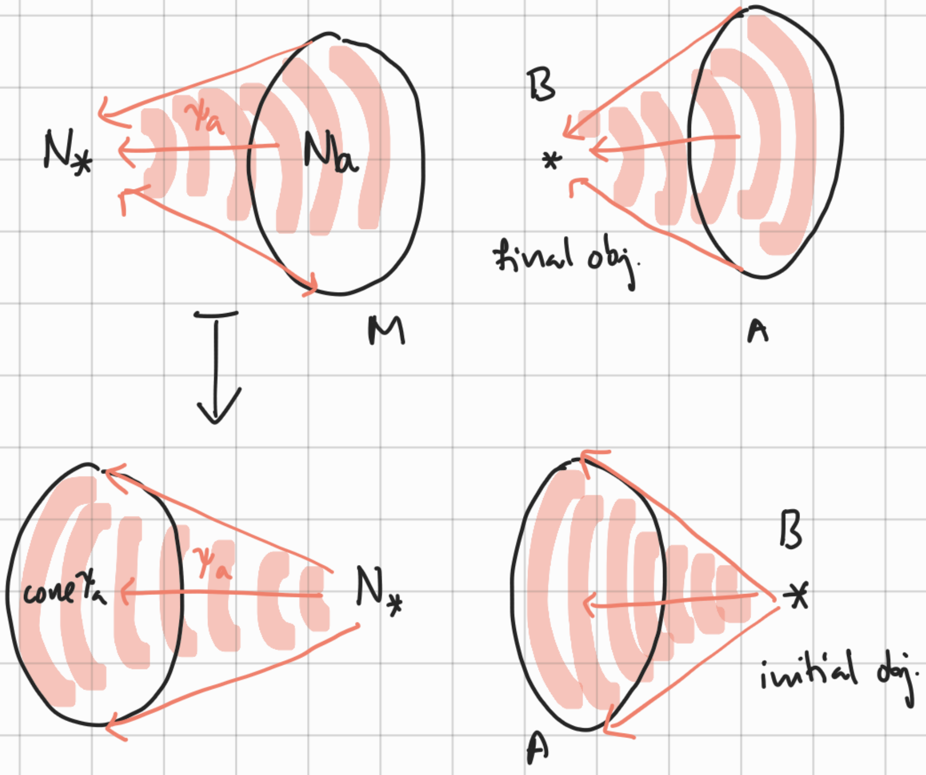
$$\mathrm{Fun}(\mathcal{B}, \mathcal{D}) = \mathrm{Fun}(\{*\}, \mathcal{D}) \simeq \mathcal{D}$$

$$\begin{aligned} F : \mathcal{D} &\longrightarrow \mathrm{Fun}(A, \mathcal{D}) \\ d &\longmapsto Fd : a \mapsto d \end{aligned}$$ constant diag. with value d

$$\mathcal{L}^*(F) = \{ (N_*, M \xrightarrow{\psi} FN_*) \mid N_* \in \mathcal{D}, \psi \text{ in } \mathrm{Fun}(A, \mathcal{D}) \}$$

$$\updownarrow \wr$$

$$\mathrm{Fun}(\textstyle\int_n f, \mathcal{D}) \ni$$



$$\downarrow \bar{S}$$

$$\mathrm{Fun}(\textstyle\int^n f, \mathcal{D}) \ni$$

$$\updownarrow \wr$$

$$\mathcal{L}_*(F) = \{ (N_*, FN_* \xrightarrow{\varphi} N) \mid N_* \in \mathcal{D} \text{ \& } \varphi \text{ in } \mathrm{Fun}(A, \mathcal{D}) \}$$

This recovers Ladkani's reflection functors.

Ladkani constructs more general reflection functors, most of which are recovered by considering the case of a single functor $f : A \to \mathcal{B}$ (althoug Ladkani works exclusively with finite posets).

§ Appendix: Proof of the Theorem

$$\begin{tikzcd}
\int_n \underline{f} & A \times \left\{ 0 \begin{smallmatrix} \nearrow 1 \\ \vdots \\ \searrow n \end{smallmatrix} \right\} \arrow[l] \\
B \arrow[u] \arrow[ur, phantom, "PO"] & \coprod_{i=1}^{n} A \arrow[l, "\underline{f}"'] \arrow[u]
\end{tikzcd}$$

$$\mathrm{Fun}(-,\mathcal{D}) \Big\downarrow$$

$$\begin{tikzcd}
\mathrm{Fun}\left(\int_n \underline{f}, \mathcal{D}\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "hPB"] & \mathrm{Fun}\left(A \times \left\{ 0 \begin{smallmatrix} \nearrow 1 \\ \vdots \\ \searrow n \end{smallmatrix} \right\}, \mathcal{D}\right) \arrow[d] \arrow[r, phantom, "\overset{\text{adjunction}}{\simeq}"] & \mathrm{Fun}\left(0 \begin{smallmatrix} \nearrow 1 \\ \vdots \\ \searrow n \end{smallmatrix}, \mathrm{Fun}(A,\mathcal{D})\right) \\
\mathrm{Fun}(B,\mathcal{D}) \arrow[r, "\underline{f}^*"'] & \mathrm{Fun}\left(\coprod_{i=1}^{n} A, \mathcal{D}\right) \arrow[r, phantom, "\simeq"] & \prod_{i=1}^{n} \mathrm{Fun}(A,\mathcal{D})
\end{tikzcd}$$

We also have a homotopy PB diagram

$$\begin{tikzcd}
\mathcal{L}^*\left(\bigoplus_{i=1}^{n}\right) \simeq \mathrm{Fun}\left(0 \begin{smallmatrix} \nearrow 1 \\ \vdots \\ \searrow n \end{smallmatrix}, \mathrm{Fun}(A,\mathcal{D})\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "hPB"] & \mathrm{Fun}(0 \to t, \mathrm{Fun}(A,\mathcal{D})) \arrow[d, "t^*"] \\
\prod_{i=1}^{n} \mathrm{Fun}(A,\mathcal{D}) \arrow[r, "\bigoplus_{i=1}^{n}"'] & \mathrm{Fun}(A,\mathcal{D})
\end{tikzcd}$$

Pasting together the above diagrams yields a homotopy PB diagram

$$\begin{tikzcd}
\mathrm{Fun}\left(\int_n \underline{f}, \mathcal{D}\right) \arrow[r] \arrow[d] \arrow[dr, phantom, "hPB"] & \mathrm{Fun}(0 \to t, \mathrm{Fun}(A,\mathcal{D})) \arrow[d, "t^*"] \\
\mathrm{Fun}(B,\mathcal{D}) \arrow[r, "F = \bigoplus_{i=1}^{n} \circ \underline{f}^*"'] & \mathrm{Fun}(A,\mathcal{D})
\end{tikzcd}$$

This shows that $\mathrm{Fun}\left(\int_n \underline{f}, \mathcal{D}\right) \xrightarrow{\sim} \mathcal{L}^*(F)$ ■

§ Appendix : Recollements

$F : \mathcal{D} \to \mathcal{C}$ exact functor between stable ∞-categories

Have canonical recollements :

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "i"] & \mathcal{L}_*(F) \arrow[l, bend right=60, "i_L"'] \arrow[l, bend left=60, "i_R"] \arrow[r, "p"] & \mathcal{D} \arrow[l, bend right=60, "p_L"'] \arrow[l, bend left=60, "p_R"]
\end{tikzcd}$$

$i(c) = (0, F(0) \to c)$

$i_L(d, Fd \xrightarrow{\varphi} c) = \text{cone}\,\varphi$

$i_R(d, Fd \to c) = c$

$p(d, Fd \to c) = d$

$p_L(d) = (d, Fd \xrightarrow{1} Fd)$

$p_R(d) = (d, Fd \to 0)$

$\text{Ker}\, p = \text{Im}(i) \qquad i_L \dashv i \dashv i_R \quad \& \quad p_L \dashv p \dashv p_R$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "j"] & \mathcal{L}^*(F) \arrow[l, bend right=60, "j_L"'] \arrow[l, bend left=60, "j_R"] \arrow[r, "q"] & \mathcal{D} \arrow[l, bend right=60, "q_L"'] \arrow[l, bend left=60, "q_R"]
\end{tikzcd}$$

$j(c) = (0, c \to F(0))$

$j_L(d, c \to Fd) = c$

$j_R(d, c \xrightarrow{\psi} Fd) = \text{cocone}\,\psi$

$q(d, c \to Fd) = d$

$q_L(d) = (d, 0 \to Fd)$

$q_R(d) = (d, Fd \xrightarrow{1} Fd)$

$\text{Ker}\, q = \text{Im}(j) \qquad j_L \dashv j \dashv j_R \quad \& \quad q_L \dashv q \dashv q_R$

Moreover, there are commutative diagrams of recollements

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "i"] \arrow[d, equal] & \mathcal{L}_*(F) \arrow[l, bend right, "i_L"'] \arrow[l, bend left, "i_R"] \arrow[r, "p"] \arrow[d] & \mathcal{D} \arrow[l, bend right, "p_L"'] \arrow[l, bend left, "p_R"] \arrow[d, "F"] \\
\mathcal{C} \arrow[r, hook, "0\to ?"] & \mathrm{Fun}(s\to t, \mathcal{C}) \arrow[l, bend right, "\mathrm{cone}"'] \arrow[l, bend left, "t^*"] \arrow[r, "s^*"] \arrow[d, phantom, sloped, "\simeq"] & \mathcal{C} \arrow[l, bend right, "{? \xrightarrow{1} ?}"'] \arrow[l, bend left, "? \to 0"] \\
& \mathcal{L}_*(\mathbb{1}_{\mathcal{C}}) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "j"] \arrow[d, equal] & \mathcal{L}^*(F) \arrow[l, bend right, "j_L"'] \arrow[l, bend left, "j_R"] \arrow[r, "q"] \arrow[d] & \mathcal{D} \arrow[l, bend right, "q_L"'] \arrow[l, bend left, "q_R"] \arrow[d, "F"] \\
\mathcal{C} \arrow[r, hook, "? \to 0"] & \mathrm{Fun}(s\to t, \mathcal{C}) \arrow[l, bend right, "s^*"'] \arrow[l, bend left, "\mathrm{cocone}"] \arrow[r, "t^*"] \arrow[d, phantom, sloped, "\simeq"] & \mathcal{C} \arrow[l, bend right, "0 \to ?"'] \arrow[l, bend left, "{? \xrightarrow{1} ?}"] \\
& \mathcal{L}^*(\mathbb{1}_{\mathcal{C}}) &
\end{tikzcd}$$

$$\cdots \dashv \underset{\Sigma \circ \mathrm{cocone}}{\underset{\rotatebox{90}{$\simeq$}}{\mathrm{cone}}} \dashv (0 \to ?) \dashv t^* \dashv (? \xrightarrow{1} ?) \dashv s^* \dashv (? \to 0) \dashv \underset{\Sigma^{-} \circ \mathrm{cone}}{\underset{\rotatebox{90}{$\simeq$}}{\mathrm{cocone}}} \dashv \cdots$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "j"] \arrow[d, "\Sigma"'] & \mathcal{L}^*(F) \arrow[l, bend right, "j_L"'] \arrow[l, bend left, "j_R"] \arrow[r, "q"] \arrow[d, "S^{-}"] & \mathcal{D} \arrow[l, bend right, "q_L"'] \arrow[l, bend left, "q_R"] \arrow[d, equal] \\
\mathcal{C} \arrow[r, hook, "i"] \arrow[d, "\Sigma^{-}"'] & \mathcal{L}_*(F) \arrow[l, bend right, "i_L"'] \arrow[l, bend left, "i_R"] \arrow[r, "p"] \arrow[d, "S^{+}"] & \mathcal{D} \arrow[l, bend right, "p_L"'] \arrow[l, bend left, "p_R"] \arrow[d, equal] \\
\mathcal{C} \arrow[r, hook, "j"] & \mathcal{L}^*(F) \arrow[l, bend right, "j_L"'] \arrow[l, bend left, "j_R"] \arrow[r, "q"] & \mathcal{D} \arrow[l, bend right, "q_L"'] \arrow[l, bend left, "q_R"]
\end{tikzcd}$$