Universal properties of derived categories (after Lurie)

$\mathcal{G}$: Grothendieck cat. (fixed)

$K(\mathrm{Inj}\,\mathcal{G}) \to \mathcal{D}(\mathcal{G}) \to \hat{\mathcal{D}}(\mathcal{G})$

Aim Characterise these as enhanced triangulated cats. with a t-structure.

§ Stable ∞-categories

$\mathcal{C}$: ∞-category

$\forall X, Y \in \mathcal{C} \rightsquigarrow \mathrm{Hom}_{\mathcal{C}}(X,Y)$ space

$ho(\mathcal{C})$: homotopy category

$\mathrm{Hom}_{\mathcal{C}}(X,Y) := \pi_0 \mathrm{Hom}_{\mathcal{C}}(X,Y)$

Def $\mathcal{C}$ is stable if

(1) $\exists\, 0 \in \mathcal{C}$: zero object:

$\forall X \in \mathcal{C}$ $\mathrm{Hom}_{\mathcal{C}}(X,0) \simeq \{*\} \simeq \mathrm{Hom}_{\mathcal{C}}(0,X)$

(2) $\forall f: X \to Y$ there exist

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "hPO"] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \quad \& \quad \begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "hPB"] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

(3) $$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$ is hPO $\Longleftrightarrow$ is hPB

Prop $\mathcal{C}$: stable $\Rightarrow$ $ho(\mathcal{C})$: tri. cat.

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd} \qquad \begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & \Sigma X \end{tikzcd}$$

suspension   loops   triangles

Def $\mathcal{T}$: triangulated cat.

An (∞-cat) **enhancement** of $\mathcal{T}$

consists of

- a stable ∞-cat $\mathcal{C}$
- an exact equiv $ho(\mathcal{C}) \xrightarrow{\sim} \mathcal{T}$

Thm All algebraic and topological

tri. cat's admit enhancements

Thm [Antieau, Lurie]

$D(\mathcal{G})$ admits a unique

enhancement $\mathcal{D}(\mathcal{G})$ and

$\mathcal{D}(\mathcal{G})$ is presentable

($\Rightarrow$ homotopy cat well gen)

$\mathcal{C}$: stable ∞-category

$t = (\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$: t-structure

Terminology:

- **smashing**: $\mathcal{C}_{\leq 0}$ closed under $\coprod$'s
- **homotopically smashing**: $\mathcal{C}_{\leq 0}$ closed under filt colimits
- **right non-deg**: $\bigcap \mathcal{C}_{\leq n} = \{0\}$
- **left non-deg**: $\bigcap \mathcal{C}_{\geq n} = \{0\}$
- **right complete**: $\mathcal{C} \xrightarrow{\sim} \lim(\cdots \to \mathcal{C}_{\geq -2} \xrightarrow{\tau_{\geq -1}} \mathcal{C}_{\geq -1} \xrightarrow{\tau_{\geq 0}} \mathcal{C}_{\geq 0})$
- **left complete**: $\mathcal{C} \xrightarrow{\sim} \underbrace{\lim(\cdots \to \mathcal{C}_{\leq 2} \xrightarrow{\tau_{\leq 1}} \mathcal{C}_{\leq 1} \xrightarrow{\tau_{\leq 0}} \mathcal{C}_{\leq 0})}_{\widehat{\mathcal{C}}}$

$\mathcal{C}^+ := \bigcup \mathcal{C}_{\leq n} \subseteq \mathcal{C}$

↑ left bounded objects

$\mathcal{C}^\heartsuit := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$: heart

↑ abelian category

Prop t : (countably) smashing

TFAE

(1) t is right non-deg.

(2) t is right complete

Aim Characterise the std. t-str on $K(\mathrm{Inj}\,\mathcal{G})$, $\mathcal{D}(\mathcal{G})$ & $\hat{\mathcal{D}}(\mathcal{G})$ in terms of the above properties.

§ Universal property of $\mathcal{D}^+(\mathcal{A})$

Thm $\mathcal{C}$: stable ∞-cat.

$t = (\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$: t-structure

Suppose that

(1) $\mathcal{C}^\heartsuit$ has enough inj

(2) t is right complete

$\Rightarrow \exists\ \mathrm{real}^+ : \mathcal{D}^+(\mathcal{C}^\heartsuit) \to \mathcal{C}$ t-ex extending $\mathcal{C}^\heartsuit \hookrightarrow \mathcal{C}$.

TFAE

(1) $\mathrm{real}^+ : \mathcal{D}^+(\mathcal{C}^\heartsuit) \xrightarrow{\sim} \mathcal{C}^+ \subseteq \mathcal{C}$

(2) $\mathrm{Inj}(\mathcal{C}^\heartsuit) \subseteq t\text{-}\mathrm{Inj}(\mathcal{C})$

$t\text{-}\mathrm{Inj}(\mathcal{C}) = (\mathcal{C}_{\leq 0}[-1])^\perp \cap \mathcal{C}_{\leq 0}$

" $\mathrm{Ext}^1_{\mathcal{C}}$-inj obj. of $\mathcal{C}_{\leq 0}$ "

Thm $\mathcal{A}$: abelian w/ enough inj.

$\mathcal{C}$: stable ∞-cat

t: right complete t-struct.

$$\Rightarrow \mathrm{Fun}^{+}(\mathcal{D}^{+}(\mathcal{A}), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{lex}(\mathcal{A}, \mathcal{C}^{\heartsuit})$$

Fun^{+} = exact functors which preserve the co-aisle & $F(\mathrm{Inj}(\mathcal{A})) \subseteq \mathcal{C}^{\heartsuit}$

Fun^{lex} = left exact functors

Rmk Proof relies on universal property of $\mathcal{D}^{+}(\mathcal{A})_{\leq 0}$ and is related to the construction of right derived functors of left exact functors.

§ Universal property of $\mathcal{D}(\mathcal{G})$

Thm $\mathcal{C}$: presentable stable ∞-cat

$t = (\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$: t-structure

$\mathcal{C}^{\heartsuit}$: Grothendieck cat. TFAE

(a) $\exists\, \mathcal{D}(\mathcal{C}^{\heartsuit}) \xrightarrow{\sim} \mathcal{C}$ t-exact extending $\mathcal{C}^{\heartsuit} \hookrightarrow \mathcal{C}$

(b) The following holds:

(1) t is non-deg

(2) t is homotopically smashing

(3) $\mathrm{Inj}(\mathcal{C}^{\heartsuit}) = t\text{-}\mathrm{Inj}(\mathcal{C})$

(4) $\forall X \in \mathcal{C}_{\geq 0}\ \exists\, C \to X$ st. $C \in \mathcal{C}^{\heartsuit}$ & $C = H_0^t(C) \twoheadrightarrow H_0^t(X)$ epi in $\mathcal{C}^{\heartsuit}$

"0-complicial"

Thm [Saorín-Stovicek, Laking, Angeleri-Marks-Vitoria, Nicolás-Saorín-Zvonareva]

$\mathcal{C}$: presentable stable ∞-cat.

t : non-deg t-structure TFAE

(1) t : smashing and $\mathcal{C}^\heartsuit$ is a Grothendieck cat.

(2) t is induced by a pure-injective cosilting obj Q

If $\mathcal{C}$ comp. gen.

(3) t is homotopically smashing.

In this case:

$\mathrm{Inj}(\mathcal{C}^\heartsuit) = t\text{-inj}(\mathcal{C}) \Leftrightarrow Q$: cotilt.

Thm $\mathcal{G}$: Grothendieck cat.

$\mathcal{C}$: presentable stable ∞-cat

t : t-structure which is

- non-deg
- homotopically smashing

& $\mathcal{C}_{\geq 0}$: aisle gen. by set of obj.

$$\Rightarrow \mathrm{LFun}^{t\text{-ex}}(\mathcal{D}(\mathcal{G}), \mathcal{C}) \xrightarrow{\simeq} \mathrm{LFun}^{ex}(\mathcal{G}, \mathcal{C}^\heartsuit)$$

Grothendieck category

LFun = colimit pres. functors

t-ex = exact functors which preserve aisle & coaisle

ex = exact functors

gen = under colimits & extensions

§ Universal property of $\widehat{\mathcal{D}}(\mathcal{G})$

$\widehat{\mathcal{C}} := \lim(\cdots \to \mathcal{C}_{\le 2} \to \mathcal{C}_{\le 1} \to \mathcal{C}_{\le 0})$

Thm $\mathcal{C}$: presentable stable ∞-cat

$\mathcal{C}^\heartsuit$: Grothendieck cat.

TFAE

(a) $\exists\, \widehat{\mathcal{D}}(\mathcal{C}^\heartsuit) \xrightarrow{\sim} \mathcal{C}$ t-exact

extending $\mathcal{C}^\heartsuit \hookrightarrow \mathcal{C}$

(b) The following holds:

(1) t is non-deg.

(2) t is homotopically smashing

(3) $\mathrm{Inj}(\mathcal{C}^\heartsuit) = t\text{-}\mathrm{Inj}(\mathcal{C})$

(4) t is left complete ($\mathcal{C} \xrightarrow{\sim} \widehat{\mathcal{C}}$)

Def [Roos] $\mathcal{G}$: Grothendieck cat.

$\mathcal{G}$ is Ab4*n ($n \geq 0$, fixed) if

$\forall I \in \mathrm{Set}$ $\forall k > n$ $R^k(\prod_I) = 0$

Def [Virili] $\mathcal{C}$: stable w/ $\prod$'

t : t-structure is n-cosmashing

if $\forall \{X_i\} \subset \mathcal{C}_{\geq 0}$ we have

$\prod X_i \in \mathcal{C}_{\geq -n}$ ($n \geq 0$, fixed)

Thm [Virili, Antieau]

(1) $\mathcal{G}$ Ab4*n $\Rightarrow$ $\mathcal{D}(\mathcal{G}) \xrightarrow{\sim} \widehat{\mathcal{D}}(\mathcal{G})$

(2) t : n-cosmashing TFAE

(a) t is left non-deg

(b) t is left complete

Always : (b) $\Rightarrow$ (a)

Coro $\mathcal{C}$: presentable stable ∞-cat

$\mathcal{C}^\heartsuit$: Grothendieck cat.

Suppose that

can remove if $\mathcal{C}$ is comp. gen. [Laking]

- t is non-deg
- t is homotopically smashing
- $\mathrm{Inj}(\mathcal{C}^\heartsuit) = t\text{-}\mathrm{Inj}(\mathcal{C})$
- t is n-cosmashing

$\Rightarrow \exists\, \mathcal{D}(\mathcal{C}^\heartsuit) \xrightarrow{\simeq} \hat{\mathcal{D}}(\mathcal{C}^\heartsuit) \xrightarrow{\sim} \mathcal{C}$

t-exact extending $\mathcal{C}^\heartsuit \hookrightarrow \mathcal{C}$

Rmk Virili uses a variant of the Coro to deduce results by Stovicek and Nicolás-Saorín-Zvonareva and extend a result of Psaroudakis-Vitoria

Thm $\mathcal{G}$: Grothendieck cat.

$\mathcal{C}$: presentable stable ∞-cat

t : t-structure which is

- right non-deg
- homotopically smashing
- left complete ($\Rightarrow$ left non-deg)

& $\mathcal{C}_{\geq 0}$: aisle gen. by set of obj.

$$\Rightarrow \mathrm{LFun}^{t\text{-ex}}(\hat{\mathcal{D}}(\mathcal{G}), \mathcal{C})$$
$$\downarrow \simeq$$
$$\mathrm{LFun}^{ex}(\mathcal{G}, \mathcal{C}^\heartsuit)$$

Grothendieck category

Rmk The above conditions hold if t is already left non-deg. & n-cosmashing for some $n \geq 0$.

§ The universal property of $\mathcal{K}(\mathrm{Inj}\, \mathcal{G})$

$\mathcal{C}$: presentable stable ∞-cat.

$t = t_{\mathcal{C}}$: t-structure. which is

- right non-deg
- homotopically smashing

& $\mathcal{C}_{\geq 0}$: aisle gen. by set of obj.

Def t is (left) anti-complete if

$\forall \mathcal{D}$ $\forall t_{\mathcal{D}}$: t-structure on $\mathcal{D}$

with the above properties

$$\begin{array}{c} \mathrm{LFun}^{t\text{-ex}}(\mathcal{C}, \mathcal{D}) \\ \downarrow \simeq \\ \mathrm{LFun}^{t\text{-ex}}(\mathcal{C}, \hat{\mathcal{D}}) \end{array}$$

Thm $\mathcal{C}$: presentable stable ∞-cat

$\mathcal{C}^{\heartsuit}$: Grothendieck cat.

TFAE

(a) $\exists\, \mathcal{K}(\mathrm{Inj}\, \mathcal{C}^{\heartsuit}) \xrightarrow{\sim} \mathcal{C}$ t-exact

extending $\mathcal{C}^{\heartsuit} \hookrightarrow \mathcal{C}$

(b) The following holds:

(1) t is right non-deg.

(2) t is homotopically smashing

(3) $\mathrm{Inj}(\mathcal{C}^{\heartsuit}) = t\text{-}\mathrm{Inj}(\mathcal{C})$

(4) t is anti-complete

Thm $\mathcal{G}$: Grothendieck cat.

$\mathcal{C}$: presentable stable ∞-cat

t: t-structure which is

- right non-deg.
- homotopically smashing

& $\mathcal{C}_{\geq 0}$: aisle gen. by set of obj.

$$\Rightarrow \mathrm{LFun}^{t\text{-ex}}(\mathcal{K}(\mathrm{Inj}\,\mathcal{G}), \mathcal{C})$$
$$\downarrow \simeq$$
$$\mathrm{LFun}^{ex}(\mathcal{G}, \mathcal{C}^{\heartsuit})$$

Grothendieck category

Q Are there "nice" conditions to ensure that a t-structure is 0-complicial or left anti-complete?

§ Summary Co-aisle $\simeq \mathcal{D}^+(\mathcal{G})$

$\mathcal{K}(\mathrm{Inj}\,\mathcal{G})$: right non-deg & anti-comp

$\mathcal{D}(\mathcal{G})$: non-deg & 0-complicial

$\hat{\mathcal{D}}(\mathcal{G})$: non-deg & left complete

$\mathrm{Inj}(\mathcal{G}) = t\text{-}\mathrm{Inj}$, t: hom. smash.

& aisle is a presentable ∞-cat.

§ References

Antieau - On the uniqueness of ∞-cat enhancements

Laking - Purity in compactly gen. ...

Lurie - Higher algebra (Ch. 1)

Lurie - Spectral alg. geo. (App. C)

Saorín-Stovicek - t-structures with...

Virili - Morita theory for stable der.