The higher Waldhausen $S_\bullet$-construction and the symplectic geometry of surfaces and their symmetric products

(based on jt. work with Dyckerhoff-Walde & with Dyckerhoff-Lekili)

Plan for today:

1 Background & Motivation

2 The higher Waldhausen $S_\bullet$-construction

3 Higher Auslander algebras of type $\mathbb{A}$

4 Fukaya categories of surfaces and their symmetric products

§ Background & Motivation

$\mathcal{A}$: enhanced triangulated category / k: field

$K(\mathcal{A})$: algebraic K-theory space

$\sim$ Waldhausen : $K(\mathcal{A}) := \Omega | S_\bullet(\mathcal{A})^{\simeq} |$

$S_\bullet(\mathcal{A})$: $S_\bullet$-construction of $\mathcal{A}$

$\forall n \geq 0$ $S_n(\mathcal{A})$: enhanced triang. cat.

face & deg $d_i : S_{n+1}(\mathcal{A}) \rightleftarrows S_n(\mathcal{A}) : s_i$

adjunctions: $d_i \dashv s_i \dashv d_{i+1}$

$t : S_n(\mathcal{A}) \xrightarrow{\sim} S_n(\mathcal{A})$ autoequivalence

$$S_n(\mathcal{A}) \simeq \mathrm{Fun}(1 \to 2 \to \cdots \to n, \mathcal{A})$$

$$\boxed{S_n(\mathrm{perf}\, k) \simeq \mathrm{perf}\, A_n}$$

$$A_n := k(1 \to 2 \to \cdots \to n)$$

$S_0(\mathcal{A}) = \{0\}$

$S_1(\mathcal{A}) = \left\{ \begin{matrix} 0 \to a_{01} \\ \downarrow \\ 0 \end{matrix} \right\} \simeq \mathcal{A} \circlearrowleft \; a \longmapsto a[1]$

$$S_2(\mathcal{A}) = \left\{ \begin{tikzcd} 0 \arrow[r] & a_{01} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & a_{02} \arrow[d] \\ & 0 \arrow[r] & a_{12} \arrow[d] \\ & & 0 \end{tikzcd} \right\} \circlearrowleft \quad \begin{tikzcd} a_{01} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & a_{02} \arrow[d] \\ 0 \arrow[r] & a_{12} \end{tikzcd} \mapsto \begin{tikzcd} a_{12} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & a_{01}[1] \arrow[d] \\ 0 \arrow[r] & a_{02}[1] \end{tikzcd}$$

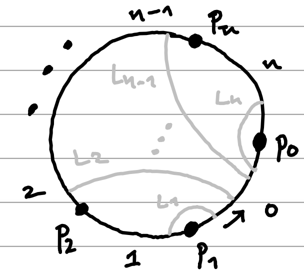
exact triangles in $\mathcal{A}$

$$S_3(\mathcal{A}) = \left\{ \begin{tikzcd} 0 \arrow[r] & a_{01} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & a_{02} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & a_{03} \arrow[d] \\ & 0 \arrow[r] & a_{12} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & a_{13} \arrow[d] \\ & & 0 \arrow[r] & a_{23} \arrow[d] \\ & & & 0 \end{tikzcd} \right\} \circlearrowleft \cdots$$

$\vdots$

exact octahedra in $\mathcal{A}$

$\mathbb{D}$: 2-dim disk $\quad \Lambda_n \subset \partial\mathbb{D}$ $n+1$ points



$\mathcal{W}(\mathbb{D}, \Lambda_n)$

Partially wrapped Fukaya category

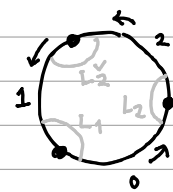
$L_1 \to L_2 \to \cdots \to L_n$

$$\boxed{\mathcal{W}(\mathbb{D}, \Lambda_n) \xleftarrow{\sim} \operatorname{perf} A_n \xrightarrow{\sim} S_n(\operatorname{perf} k)}$$

- $\operatorname{thick} \bigoplus L_i = \mathcal{W}(\mathbb{D}, \Lambda_n)$
- $\mathbb{R}\mathrm{End}(\bigoplus L_i) \simeq A_n$

derived Morita theory [Rickard-Keller]

e.g.



$\underbrace{\mathbb{R}\mathrm{End}(L_1 \oplus L_2 \oplus L_2^\vee)}_{A_\infty\text{-algebra}}$

$$\begin{tikzcd} L_1 \arrow[r, "x"] & L_2 \arrow[d, "x"] \\ & L_2^\vee \arrow[ul, "y"] \end{tikzcd}$$

$\mu_3(y, x, x) = \mathrm{id}$

$\mu_3(x, x, y) = \mathrm{id}$

$|x| = 0, \; |y| = 1 \qquad \mu_3(x, y, x) = \mathrm{id}$

§ The higher Waldhausen $S_\bullet$-construction

Hesselholt-Madsen, Dyckerhoff, Poguntke

$d \geq 1 \rightsquigarrow S_\bullet^{(d)}(\mathcal{A})$: d-dim $S_\bullet$-construction

<u>Poguntke</u> $K(\mathcal{A}) \simeq \Omega^d \,|\, S_\bullet^{(d)}(\mathcal{A})^{\simeq} \,|$

$$S_0^{(2)}(\mathcal{A}) \simeq \{0\}$$

$$S_1^{(2)}(\mathcal{A}) \simeq \{0\}$$

$$S_2^{(2)}(\mathcal{A}) \simeq \mathcal{A}$$

$$S_3^{(2)}(\mathcal{A}) \simeq \left\{ \begin{tikzcd} & a_{012} \arrow[rr] \arrow[dl] \arrow[dd] & & a_{013} \arrow[dl] \arrow[dd] \\ 0 \arrow[rr] \arrow[dd] & & 0 \arrow[dd] & \\ & 0 \arrow[rr] \arrow[dl] & & a_{023} \arrow[dl] \\ 0 \arrow[rr] & & a_{123} & \end{tikzcd} \right\}$$

bicart. cube

$\vdots$

exact complex with 4 terms

Nistor, Fiedorowicz-Loday, Getzler-Jones

Λ : paracyclic category

$\operatorname{ob} \Lambda := \{ \underline{n} \mid n \geq 0 \}$

$$\underbrace{\{ \sigma : \mathbb{Z} \xrightarrow{\leq} \mathbb{Z} \mid \forall i \ \sigma(i+m+1) = \sigma(i) + n + 1 \}}_{\Lambda(\underline{m}, \underline{n})}$$

$$\operatorname{Aut}(\underline{n}) = \langle t : i \mapsto i+1 \rangle \cong \mathbb{Z}$$

$\Lambda(\underline{m}, \underline{n})$ is a poset w.r.t.

$$\sigma \leq \tau \overset{\text{def}}{\iff} \forall i \in \mathbb{Z} : \sigma(i) \leq \tau(i)$$

$$\Delta([m],[n]) = \{ \sigma \mid 0 \leq \sigma(0) \leq \sigma(1) \leq \cdots \leq \sigma(m) \leq n \}$$

$$\rightsquigarrow \Delta([m],[n]) \overset{\leq}{\hookrightarrow} \Lambda(\underline{m}, \underline{n})$$

$$\rightsquigarrow \Delta \longrightarrow \Lambda, \quad [n] \longmapsto \underline{n}$$

Example $\Lambda(\underline{1},\underline{n})$ $\quad \square = \deg$

$$\begin{tikzcd}[column sep=small]
 & & & & & & \cdots 0,n+1,n+1,\cdots \arrow[d, phantom, "\parallel"] & \\
\boxed{00} \arrow[r] & 01 \arrow[r] \arrow[d] & 02 \arrow[r] \arrow[d] & 03 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0n \arrow[r] \arrow[d] & \boxed{0,n+1} \arrow[d] & \\
 & \boxed{11} \arrow[r] & 12 \arrow[r] \arrow[d] & 13 \arrow[r] \arrow[d] & \cdots \arrow[r] & 1n \arrow[r] \arrow[d] & \boxed{1,n+1} \arrow[d] & \\
 & & \boxed{22} \arrow[r] & 23 \arrow[r] & \cdots \arrow[r] & 2n \arrow[r] & 2,n+1 \arrow[r] & \boxed{2,n+2}
\end{tikzcd}$$

$t \searrow$

$\vdots$

Def $\sigma \in \Lambda(\underline{m},\underline{n})$ is deg if $\sigma : \mathbb{Z} \xrightarrow{\leq} \mathbb{Z}$ is not injective

Example $\Lambda(\underline{2},\underline{3})$ $\quad \square = \deg$.

$$\begin{tikzcd}[column sep=small]
 & 012 \arrow[rr] \arrow[dl] \arrow[dd] & & 013 \arrow[dl] \arrow[dd] \\
\boxed{112} \arrow[rr] \arrow[dd] & & \boxed{113} \arrow[dd] & \\
 & \boxed{022} \arrow[rr] \arrow[dl] & & 023 \arrow[dl] \\
\boxed{122} \arrow[rr] & & 123 &
\end{tikzcd}$$

$[012, t\cdot 023]$ closed interval

$t \searrow$

Def $S_n^{(d)}(\mathcal{A}) := \mathrm{Fun}_*^{ex}(\Lambda(\underline{d},\underline{n}),\mathcal{A})$

$a \in \mathrm{Fun}_* \overset{def}{\Longleftrightarrow} \forall \sigma : \deg \quad a_\sigma \simeq 0$

$a \in \mathrm{Fun}^{ex} \overset{def}{\Longleftrightarrow} \forall \sigma :$ non-deg the $(d+1)$-cube

$$[\sigma, t\cdot\sigma] \hookrightarrow \Lambda(\underline{d},\underline{n}) \xrightarrow{a} \mathcal{A}$$

is (homotopy) bicartesian.

$\rightsquigarrow S_\bullet^{(d)}(\mathcal{A})$: paracyclic enhanced triangulated category

$\forall n < d \quad S_n^{(d)}(\mathcal{A}) \simeq 0$

$n = d \quad S_d^{(d)}(\mathcal{A}) \simeq \mathcal{A}$

$S_{d+1}^{(d)}(\mathcal{A}) \simeq$ exact complexes with $d+1$ terms

$P(\underline{d},\underline{m}) := \{\sigma \mid \sigma(0) = 0\} \subset \Lambda(\underline{d},\underline{n})$

Prop [Dyckerhoff - J - Walde]

$\Longrightarrow S_n^{(d)}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Fun}_*(P(\underline{d},\underline{n}),\mathcal{A})$

§ Higher Auslander algebras of type $\mathbb{A}$

k: comm. ring $(n \geq, d \geq 1)$

$A_n^{(d)}$: higher Auslander alg of type $\mathbb{A}$

Def [Iyama, Oppermann–Thomas]

obj: $I = \{i_1 < \cdots < i_d\} \subseteq \{1, 2, \dots, n\}$

$$A_n^{(d)}(I, J) = \begin{cases} k \cdot f_{JI} & I \rightsquigarrow J \\ 0 & \text{otherwise} \end{cases}$$

$I \rightsquigarrow J \overset{\text{def}}{\iff} i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_d \leq j_d$

Composition: $f_{KJ} \circ f_{JI} = f_{KI}$

Example $A_n^{(1)} = k(1 \to 2 \to \cdots \to n)$

$$A_4^{(2)} = \begin{tikzcd} 12 \arrow[r, "f_{13,12}"] & 13 \arrow[r] \arrow[d] & 14 \arrow[d] \\ & 23 \arrow[r] & 24 \arrow[d] \\ & & 34 \end{tikzcd} \sim \begin{tikzcd} 012 \arrow[r] \arrow[d] & 013 \arrow[r] \arrow[d] & 014 \arrow[d] \\ \boxed{022} \arrow[r] & 023 \arrow[r] \arrow[d] & 024 \arrow[d] \\ & \boxed{033} \arrow[r] & 034 \end{tikzcd}$$

$^1\pi_2 \times^2 \pi_3$

$\mathcal{A}$: A_∞-category / dg category

Prop [DJW] $S_n^{(d)}(\mathcal{A}) \simeq A_\infty\text{-Fun}(A_n^{(2)}, \mathcal{A})$

$$\boxed{S_n^{(d)}(\operatorname{perf} k) \xrightarrow{\sim} \operatorname{perf} A_n^{(d)}}$$

Digression k: field $D := \operatorname{Hom}_k(-, k)$

- $n > d \Rightarrow \operatorname{gl.dim} A_n^{(d)} = d$
- $\mathbb{S} := - \overset{\mathbb{L}}{\underset{A_n^{(d)}}{\otimes}} DA_n^{(d)} : \operatorname{perf} A_n^{(d)} \xrightarrow{\sim} \operatorname{perf} A_n^{(d)}$
- Serre duality: $D\operatorname{Hom}(X, Y) \cong \operatorname{Hom}(Y, \mathbb{S}X)$
- $t \curvearrowright S_n^{(d)}(\operatorname{perf} k) \simeq \operatorname{perf} A_n^{(d)} \circlearrowleft \mathbb{S}_d$

  $\mathbb{S}_d := \mathbb{S}[-d]$ d-Auslander-Reiten transl.
- $U(A_n^d) = \{\mathbb{S}_d^i(A_n^{(d)}) \mid i \in \mathbb{Z}\} \subset \operatorname{perf} A_n^{(d)}$

$$S_n^{(d)}(\mathcal{A}) \underset{[DJW]}{=} A_\infty\text{-Fun}(\underset{A_\infty\text{-cat}}{U(A_n^d)}, \mathcal{A})$$

Fukaya categories of surfaces and their symmetric products

Σ : surface, $\Lambda \overset{\text{fin}}{\subset} \partial\Sigma \neq \emptyset$ stops

$\mathrm{Sym}^d \Sigma := \underbrace{\Sigma \times \Sigma \times \cdots \times \Sigma}_{d \text{ times}} / \mathfrak{S}_d$

$\Lambda^{(d)} := \bigcup_{p \in \Lambda} p + \mathrm{Sym}^{d-1}\Sigma \subset \mathrm{Sym}^d \Sigma$

Auroux $\mathcal{W}(\mathrm{Sym}^d \Sigma, \Lambda^{(d)})$

partially wrapped Fukaya category

Grading issues

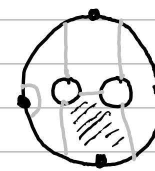
$(d=1)$ Multiple $\mathbb{Z}$-gradings

$(d \geq 2)$ genus $\Sigma \geq 1 \Rightarrow$ only $\mathbb{Z}/2$-grading.

$\Sigma = \mathbb{D}$ & $d \geq 1$: ess. unique $\mathbb{Z}$-grading.

$L_1, \dots, L_n \subset \Sigma \setminus \Lambda$ arcs with ends in $\partial\Sigma$

- $\forall i \neq j \quad L_i \cap L_j = \emptyset$
- $\Sigma \setminus \bigcup L_i$: disjoint union of disk each with at most one stop



$I = \{i_1 < i_2 < \cdots < i_d\} \subseteq \{1, 2, \dots, n\}$

$\rightsquigarrow L_I := \prod_{i \in I} L_i \subset \mathrm{Sym}^d \Sigma$

$\rightsquigarrow T = \bigoplus_I L_I \in \mathcal{W}(\mathrm{Sym}^d \Sigma, \Lambda^{(d)})$

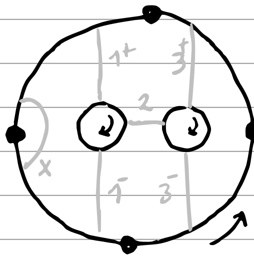
Thm [Auroux] TFSH

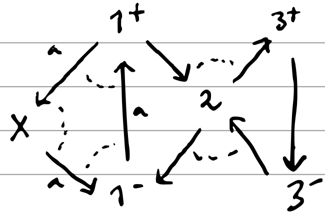
(1) $\mathrm{perf}\, \mathbb{R}\mathrm{End}(T) \xrightarrow{\simeq} \mathcal{W}(\mathrm{Sym}^d \Sigma, \Lambda^{(d)})$

(2) $\mathbb{R}\mathrm{End}(T)$ = strands algebra of Lipshitz-Ozsváth-Thurston

(3) If each disk in $\Sigma \setminus \bigcup L_i$ contains exactly one stop, then $\mathbb{R}\mathrm{End}(T)$ is a dga (rather than A_∞)

Example $(d=1)$

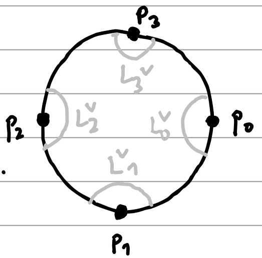




$$\mu_3(a,a,a) = id$$

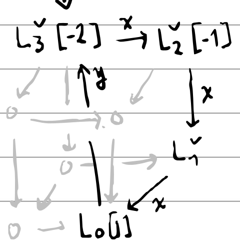
$\rightsquigarrow$ see Dyckerhoff-Kapranov and Haiden-Katzarkov-Kontsevich

Kontsevich



generalises to $3 \mapsto n \geq 1$

due to grading conv.



$$\mu_4(y,x,x,x) = id$$

$$\mu_4(x,x,x,y) = id$$

$$\mu_4(x,x,y,x) = id$$

$$\mu_4(x,y,x,x) = id$$

$$|x| = 0, \quad |y| = 2$$

$$\boxed{W(\mathbb{D}, \Lambda_n) \xleftarrow{\sim} \operatorname{perf} A_n^{(n-1)} \xrightarrow{\sim} S_n^{(n-1)}(\mathcal{A})}$$

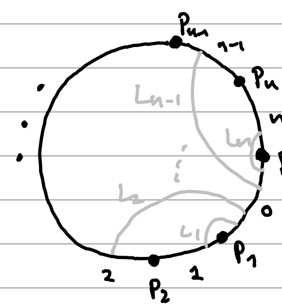
$$A_n^{(n-1)} \simeq k(n \xrightarrow{x} \cdots \xrightarrow{x} 2 \xrightarrow{x} 1)/(x^2)$$

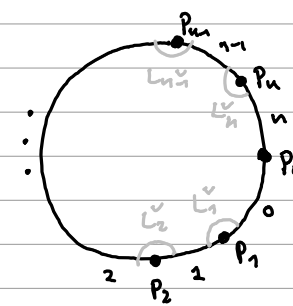
$$\mathbb{C}\text{-}\dim \mathbb{D} = 1 \neq n-1 = \operatorname{gldim} A_n^{(n-1)}$$

$\rightsquigarrow$ $\operatorname{Sym}^d \mathbb{D} \cong 2d$-dim ball

Thm [Dyckerhoff-J-Lekili]

$$\begin{tikzcd}
 & W(\operatorname{Sym}^d \mathbb{D}, \Lambda_n^{(d)}) & \\
\operatorname{perf} A_n^{(d)} \arrow[ur, "\simeq"] \arrow[dr, "\simeq"'] \arrow[rr, dashed, leftrightarrow, "{[\text{Beckert}]}"'] & & \operatorname{perf} A_n^{(n-d)} \arrow[ul, "\simeq"'] \arrow[dl, "\simeq"] \\
 & W(\operatorname{Sym}^{n-d} \mathbb{D}, \Lambda_n^{(n-d)}) &
\end{tikzcd}$$





Easy $\mathbb{R}\mathrm{End}\left(\bigoplus_{I} L_I^\vee[-\mathrm{rk}\, I]\right) \cong A_n^{(n-d)}$

$\rightsquigarrow \mathrm{perf}\, A_n^{(n-d)} \xrightarrow{\simeq} \mathcal{W}(\mathrm{Sym}^d \mathbb{D}, \Lambda_n^{(d)})$

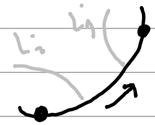
We focus on $\mathbb{R}\mathrm{End}\left(\bigoplus_{I} L_I\right) \simeq A_n^{(d)}$

Auroux (w/o grading)

$$\mathrm{Hom}(L_I, L_J) \cong \bigoplus_{\pi \in \mathfrak{S}_d} \underbrace{\bigotimes_{a=1}^{d} \mathrm{Hom}(L_{i_a}, L_{j_{\pi(a)}})}_{\mathrm{Hom}(L_I, L_J)^\pi}$$

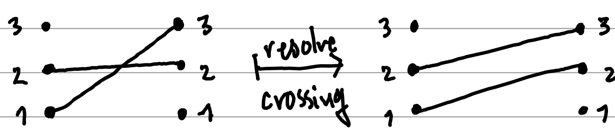
$\mathrm{Hom}(L_I, L_J)^\pi$ in degree $-\mathrm{inv}(\pi)$

$$\mathrm{Hom}(L_i, L_j) \cong \begin{cases} \Bbbk & i \leq j \\ 0 & i > j \end{cases}$$



Example $\mathrm{Hom}(L_{12}, L_{23})$ $(d=2, n \geq 3)$

$$\mathrm{Hom}(L_1, L_3) \otimes \mathrm{Hom}(L_2, L_2) \xrightarrow{\partial} \mathrm{Hom}(L_1, L_2) \otimes \mathrm{Hom}(L_1, L_3)$$



$\mathcal{B}_d$: Bruhat order on $\mathfrak{S}_d$

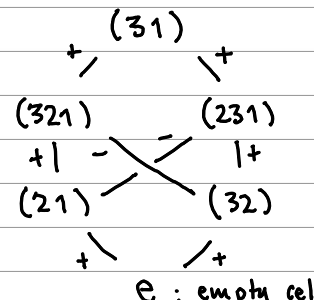
$\pi' \lhd \pi \overset{\text{def}}{\Longleftrightarrow} \mathrm{inv}(\pi) = 1 + \mathrm{inv}(\pi')$ and $\pi = s\pi'$ s: simple transp.

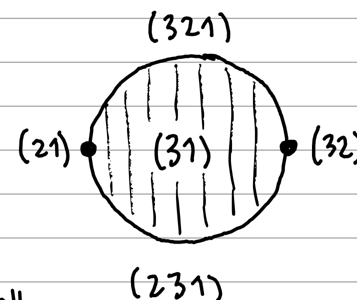
covering relation

Thm [Björner-Wachs]

Each closed interval in $\mathcal{B}_d$ is the incidence poset of a regular CW cpx homeomorphic to a ball or a point.

Example $\mathcal{B}_3 = [e, (31)]$





Reduced cellular homology complex:

$$\Bbbk(31) \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \Bbbk(321) \oplus \Bbbk(231) \xrightarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} \Bbbk(21) \oplus \Bbbk(32) \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} \Bbbk \cdot e$$

acyclic complex!

$I \leq J \overset{def}{\iff} \forall a: i_a \leq j_a$

Prop [DJL]

uniquely det. by I, J

$$\mathrm{Hom}(L_I, L_J) \cong \begin{cases} C([e, \pi_0^{JI}]) & I \leq J \\ 0 & I \not\leq J \end{cases}$$

deg 0

$$\rightsquigarrow H^*\mathrm{Hom}(L_I, L_J) \cong \begin{cases} k(0) & I \rightsquigarrow J \\ 0 & \text{otherwise} \end{cases}$$

Recall:

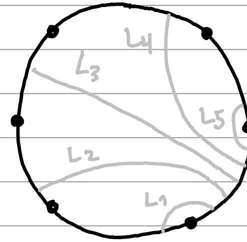
$$I \rightsquigarrow J \overset{def}{\iff} i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_d \leq j_d$$

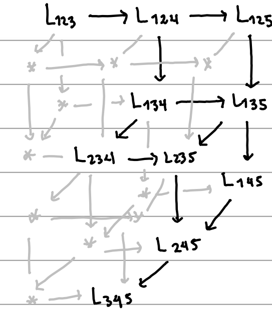
$$\Rightarrow \mathbb{R}\mathrm{End}(T) \xrightarrow[\text{q iso}]{\sim} H^0(\mathbb{R}\mathrm{End}(T))$$

$$\& \quad H^0(\mathbb{R}\mathrm{End}(T)) \cong A_n^{(d)}$$

$$\boxed{\mathrm{perf}\, A_n^{(d)} \xrightarrow{\sim} \mathcal{W}(\mathrm{Sym}^d \mathbb{D}, \Lambda_n^{(d)})}$$

Example

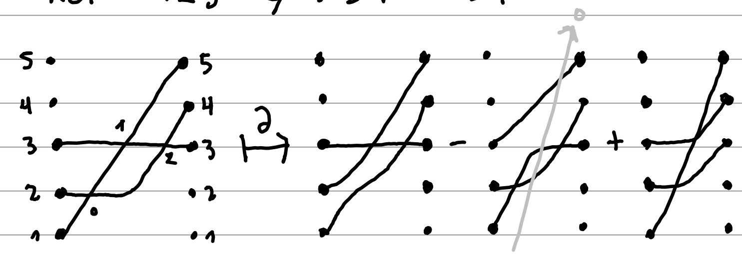




$A_5^{(3)}$

$$\mathrm{Hom}(L_{123}, L_{345}) \cong C([e, (13)]) = C(\mathcal{B}_3)$$

Note: $123 \not\rightsquigarrow 345 \quad (3 \not\leq 2)$



$$\mathrm{Hom}(L_{123}, L_{345})^{(31)} \xrightarrow[\partial]{} \mathrm{Hom}(L_{123}, L_{345})^{(321)} \oplus \mathrm{Hom}(L_{123}, L_{345})^{(231)}$$

$$\underbrace{k(31) \xrightarrow{\binom{1}{1}} k(321) \oplus k(231) \xrightarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} k(21) \oplus k(32) \xrightarrow{(1\ \ 1)} k \cdot e}$$

$$C(\mathcal{B}_3) = C([e, (31)])$$

Coro $\mathcal{W}(\mathrm{Sym}^d \mathbb{D}, \Lambda^{(d)}_\bullet) \simeq S^{(d)}_\bullet(\mathrm{perf}\, k)$

Tanaka Coherence data for $\mathcal{W}(\mathbb{D}, \Lambda_\bullet)$ in symplectic terms.

Stop removal functor

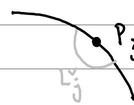
$\mathcal{W}^{(d)}_n := \mathcal{W}(\mathrm{Sym}^d \mathbb{D}, \Lambda^{(d)}_n)$

$d_j : \mathcal{W}^{(d)}_{n+1} \twoheadrightarrow \mathcal{W}^{(d)}_n$

Verdier quotient at ess. image of

Orlov functor $\mathcal{W}^{(d-1)}_n \hookrightarrow \mathcal{W}^{(d)}_{n+1}$

induced by $L_I \mapsto L_j^\vee \times L_I$



Paracyclic shift $\tau[d] \circlearrowright \mathcal{W}(\mathrm{Sym}^d \mathbb{D}, \Lambda^{(d)}_n)$

τ : rotation $\circlearrowleft$ by $2\pi/n+1$

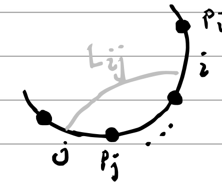
$\tau^{n+1} \simeq [d(n-d)]$ "fractionally CY"

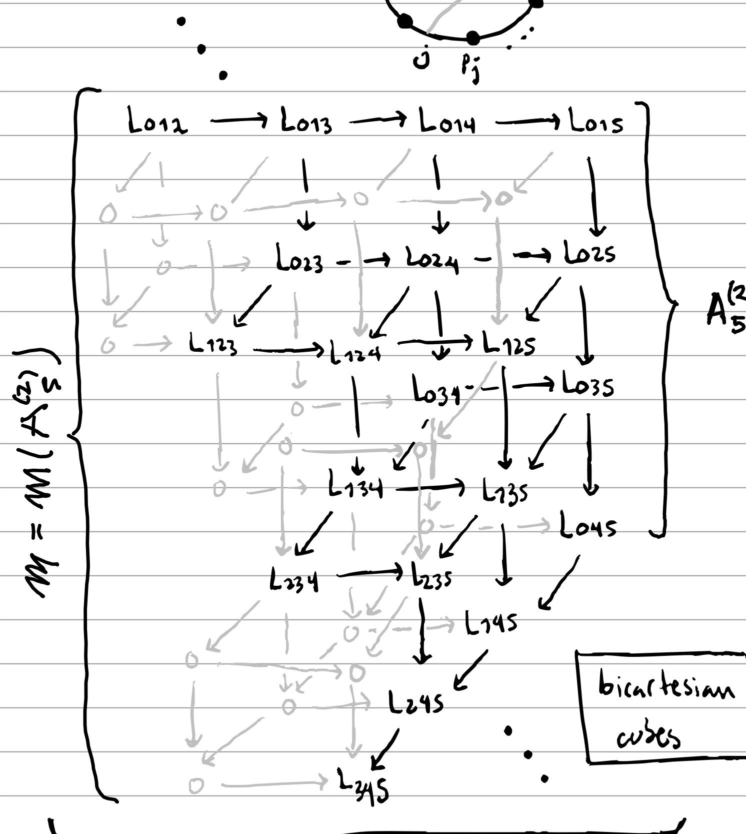
$\quad\quad ||$

$[(n-d)d]$ $\quad \mathcal{W}^{(d)}_n \simeq \mathcal{W}^{(n-d)}_n$

Example $\mathcal{W}(\mathrm{Sym}^2 \mathbb{D}, \Lambda^{(2)}_5)$

$L_{ijk} := L_{ij} \times L_{ik}$





$\bigvee_{i \in \mathbb{Z}} \mathcal{M}(A^{(2)}_5)[di] \overset{\text{Iyama}}{\simeq} \mathcal{U}(A^{(2)}_5) \hookrightarrow \mathcal{W}(\mathrm{Sym}^2 \mathbb{D}, \Lambda^{(2)}_5)$