Partially wrapped Fukaya categories of symmetric products of marked disks

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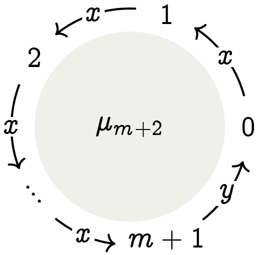
$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
\iddots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots
\end{array}
$$

Exact $(m+2)$-angles in A_∞-categories (Kontsevich)

$\mathcal{C}_m$ $(m \geq -1)$: A_∞-category



- $|x| = 0$ and $|y| = m$
- $\mu_{m+2}(x, \ldots, x, y, x, \ldots, x) = 1$
- $\mu_{\neq m+2} = 0$ (mod unitality)

Why are these interesting?

$\mathrm{Fun}_{A_\infty}(\mathcal{C}_m, \mathcal{A})$: exact $(m+2)$-angles

$\mathcal{A}$: A_∞-category

$\mathcal{C}_{-1} \to \mathcal{A} \rightsquigarrow$ zero object

$\mathcal{C}_0 \to \mathcal{A} \rightsquigarrow$ isomorphism

$\mathcal{C}_1 \to \mathcal{A} \rightsquigarrow$ exact triangle

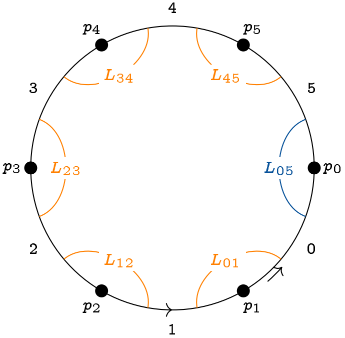
$\vdots$

$\mathcal{C}_m \to \mathcal{A} \rightsquigarrow$ exact $(m+2)$-angle

Symplectic interpretation of exact $(m+2)$-angles

Partially wrapped Fukaya category

$$\mathcal{W}(\mathbb{D}, \Lambda_n)$$



$$\mathrm{thick}\left(\bigoplus_{i=1}^{n} L_{i,i+1}\right) = \mathcal{W}(\mathbb{D}, \Lambda_n)$$

There exist **grading structures** such that

$$\mathcal{C}_{n-1} \simeq \mathrm{REnd}\left(L_{0n} \oplus \bigoplus_{i=1}^{n} L_{i,i+1}\right)$$

Moreover $L_{0n} \in \mathrm{thick}\left(\bigoplus_{i=1}^{n} L_{i,i+1}\right)$ and

$$\mathrm{REnd}\left(\bigoplus_{i=1}^{n} L_{i,i+1}\right) \cong \mathbf{k}\vec{A}_n / J^2$$

where $\vec{A}_n = 1 \to 2 \to \cdots \to n$

Theorem (Folklore)

$$\mathrm{perf}(\mathbf{k}\vec{A}_n / J^2) \xrightarrow{\sim} \mathcal{W}(\mathbb{D}, \Lambda_n)$$

Auroux, Kontsevich, Seidel, ...

Higher octahedra in A_∞-categories (Hermes)

A_∞-category $\mathcal{M}_n$

$$\begin{tikzcd}
01 \arrow[r] & 02 \arrow[r] \arrow[d] & 03 \arrow[r] \arrow[d] & 04 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & 0n \arrow[d] \\
& 12 \arrow[r] \arrow[ul, "\tau"] & 13 \arrow[r] \arrow[d] \arrow[ul, "\tau"] & 14 \arrow[r] \arrow[d] \arrow[ul, "\tau"] & \cdots \arrow[r] \arrow[d] \arrow[ul, "\tau"] & 1n \arrow[d] \arrow[ul, "\tau"] \\
& & 23 \arrow[r] \arrow[ul, "\tau"] & 24 \arrow[r] \arrow[d] \arrow[ul, "\tau"] & \cdots \arrow[r] \arrow[d] \arrow[ul, "\tau"] & 2n \arrow[d] \arrow[ul, "\tau"] \\
& & & 34 \arrow[r] \arrow[ul, "\tau"] & \cdots \arrow[r] \arrow[d] \arrow[ul, "\tau"] & 3n \arrow[d] \arrow[ul, "\tau"] \\
& & & & \ddots \arrow[r] \arrow[ul, "\tau"] & \vdots \arrow[d] \arrow[ul, "\tau"] \\
& & & & & n-1,n \arrow[ul, "\tau"]
\end{tikzcd}$$

$$|\tau| = 1$$

For all $0 \leq i < j \leq n$ the diagram

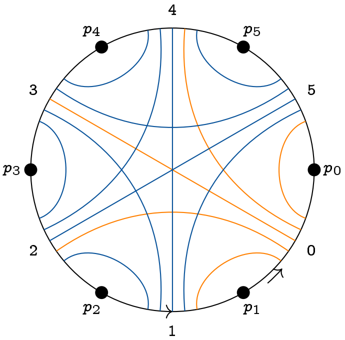
$$\begin{tikzcd}
& (i,j+1) \oplus (i+1,j) \arrow[dd] \\
(i,j) \arrow[ur] \arrow[r, phantom, "\mu_3"] & \\
& (i+1,j+1) \arrow[uul, phantom] \arrow[ul, "\tau"]
\end{tikzcd}$$

is an exact triangle in $\operatorname{perf}(\mathcal{M}_n)$

$$\mathcal{M}_n \to \mathcal{A} \rightsquigarrow n\text{-octahedron}$$

Symplectic interpretation of higher octahedra

$$\mathcal{W}(\mathbb{D}, \Lambda_n)$$



$$\mathrm{thick}\left(\bigoplus_{i=1}^{n} L_{0i}\right) = \mathcal{W}(\mathbb{D}, \Lambda_n)$$

There exist **grading structures** such that

$$\mathcal{M}_n \simeq \mathrm{REnd}\left(\bigoplus_{i<j} L_{ij}\right)$$

Moreover ($L_i := L_{0i}$)

$$\mathrm{REnd}\left(\bigoplus_{i=1}^{n} L_i\right) \cong \mathbf{k}\vec{A}_n$$

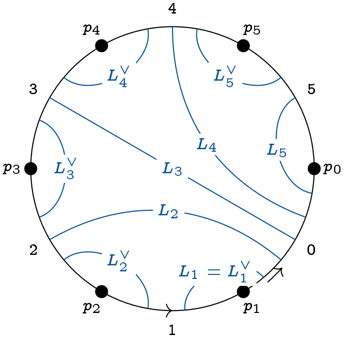
Theorem (Folklore)

$$\mathrm{perf}(\mathbf{k}\vec{A}_n) \xrightarrow{\sim} \mathcal{W}(\mathbb{D}, \Lambda_n)$$

Auroux, Kontsevich, Seidel, ...

Koszul duality for the path algebra $\vec{A}_n$

$$\mathcal{W}(\mathbb{D}, \Lambda_n)$$



$\mathrm{REnd}(\bigoplus_{i=1}^n L_i)$ is a **Koszul A_∞-algebra**

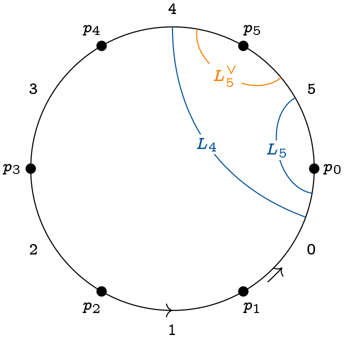
$$A_n := \mathrm{REnd}(\textstyle\bigoplus_{i=1}^n L_i)$$
$$A_n^\vee := \mathrm{REnd}(\textstyle\bigoplus_{i=1}^n L_i^\vee)$$

Koszul duality (Keller)

$$\begin{tikzcd}
\mathrm{perf}(A_n) \arrow[rr, leftrightarrow, "\text{Koszul Duality}"] \arrow[dr, "\simeq"'] & & \mathrm{perf}(A_n^\vee) \arrow[dl, "\simeq"] \\
& \mathcal{W}(\mathbb{D}, \Lambda_n) &
\end{tikzcd}$$

Stop removal functors & recollements (Auroux, Sylvan)

Identify $\Lambda_n = \Lambda_{n+1} \setminus \{p_{n+1}\}$



Stop-removal functor

$$s : \mathcal{W}(\mathbb{D}, \Lambda_{n+1}) \to \mathcal{W}(\mathbb{D}, \Lambda_n)$$

$$\begin{tikzcd} & L_{n+1} \arrow[dd] \\ L_n \arrow[ur] & \\ & L_{n+1}^\vee \arrow[ul, "{[1]}"] \end{tikzcd} \quad \xmapsto{s} \quad \begin{tikzcd} & s(L_{n+1}) \arrow[dd] \\ s(L_n) \arrow[ur, "\simeq"] & \\ & \cancelto{0}{s(L_{n+1}^\vee)} \arrow[ul, "{[1]}"] \end{tikzcd}$$

$$\operatorname{thick}(L_{n+1}^\vee) \begin{array}{c} \xleftarrow{\iota_L} \\ \xrightarrow{\iota} \\ \xleftarrow[\iota_R]{} \end{array} \mathcal{W}(\mathbb{D}, \Lambda_{n+1}) \begin{array}{c} \xleftarrow{s_L} \\ \xrightarrow{s} \\ \xleftarrow[s_R]{} \end{array} \mathcal{W}(\mathbb{D}, \Lambda_n)$$

$$\iota_L \dashv \iota \dashv \iota_R \qquad \iota = \ker(s) \qquad s_L \dashv s \dashv s_R$$

The Waldhausen $S_\bullet$-construction

Stop-removal functors and their adjoints turn

$$\mathcal{W}(\mathbb{D},\Lambda_n)\simeq \operatorname{perf}(\mathcal{M}_n)\qquad n\geq 0,$$

into a **simplicial** triangulated A_∞-category

- Coherence is established *combinatorially*
- [Tanaka] Approach via *stack of broken paracycles*

Theorem (Folklore)

$\mathcal{A}=\operatorname{perf}(A)$

$$\mathcal{W}(\mathbb{D},\Lambda_n)\otimes\mathcal{A}\simeq S_n(\mathcal{A})\qquad n\geq 0$$

$S_\bullet(\mathcal{A})$: ***Waldhausen $S_\bullet$-construction***

Rotating the disk $\rightsquigarrow$ **paracyclic** structure

K-theory of A_∞-categories (Waldhausen, Thomason, ...)

Definition

$\mathcal{A}$: triangulated A_∞-category

$$K(\mathcal{A}) := \Omega|S_\bullet(\mathcal{A})^{\simeq}|$$

Algebraic K-theory space of $\mathcal{A}$

$$K_m(\mathcal{A}) := \pi_m(K(\mathcal{A}), 0)$$

m-th algebraic K-group of $\mathcal{A}$

$K_0(\mathcal{A})$: **Grothendieck group** of $\mathcal{A}$

- base point: $0 \in \mathcal{A}$: zero object
- loop at 0: object $0 \xrightarrow{A} 0$ in $\mathcal{A}$
- 2-simplex:

$$\begin{tikzcd} & 0 \arrow[dr, "A_{12}"] & \\ 0 \arrow[ur, "A_{01}"] \arrow[rr, "A_{02}"'] & & 0 \end{tikzcd}$$

$$A_{01} \to A_{02} \to A_{12} \to A_{01}[1]$$
$$\rightsquigarrow [A_{01}] + [A_{12}] = [A_{02}]$$

- 3-simplex: octahedron in $\mathcal{A}$

Partially wrapped Fukaya categories (Auroux, Sylvan)

Σ : Riemann surface, $\partial\Sigma \neq \emptyset$
$\Lambda \subset \partial\Sigma$: finite set of marked points

$$\mathrm{Sym}^d(\Sigma) = \underbrace{\Sigma \times \cdots \times \Sigma}_{d \text{ times}} / \mathfrak{S}_d$$

$$\Lambda^{(d)} = \bigcup_{p \in \Lambda} \{p\} \times \mathrm{Sym}^{d-1}(\Sigma)$$

$\mathcal{W}(\mathrm{Sym}^d(\Sigma), \Lambda^{(d)})$
partially wrapped Fukaya category

Proposition (Perutz)

$\exists\omega$ symplectic form on $\mathrm{Sym}^d(\Sigma)$
s.t. $\omega = \omega_\Sigma^{\times d}$ away from big diagonal

- $L_1, \ldots, L_d$: Lagrangians in Σ
- $\forall i \neq j : L_i \cap L_j = \emptyset$

$\Rightarrow \prod L_i \in \mathrm{Sym}^d(\Sigma)$: Lagrangian

Kontsevich's proposal

$\mathcal{W}(\mathrm{Sym}^d(\Sigma), \Lambda^{(d)}) \simeq$ homotopy colimit of triangulated A_∞-categories of the form

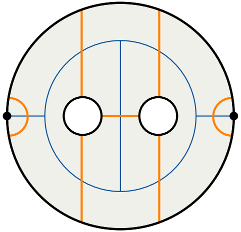
$$\bigotimes_i \mathcal{W}(\mathrm{Sym}^{d_i}(\mathbb{D}), \Lambda_{n_i}^{(d_i)}) : \sum_i d_i = d$$

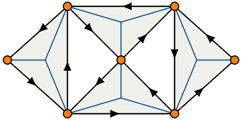
where

$$\mathrm{perf}(A) \otimes \mathrm{perf}(B) = \mathrm{perf}(A \otimes B)$$

Haiden-Katzarkov-Kontsevich
Dyckerhoff-Kapranov

Complete proofs in the case $d = 1$





$\mathrm{REnd}(\bigoplus L_i)$

shaded triangle $\rightsquigarrow$ exact triangle (μ_3)
empty triangle $\rightsquigarrow$ no relations

Work in progress (DJL)

Gluing description for genus$(\Sigma) = 0$ & $d \geq 1$

- [Lekili-Polishchuk] Direct computation for genus$(\Sigma) = 0$ & $d \geq 1$
- genus$(\Sigma) > 0 \Rightarrow$ no $\mathbb{Z}$-grading!
- precise description of indexing diagram (difficult)

$\mathbb{D}$: 2-dim unit disk

$\Lambda_n \subset \partial\mathbb{D}$: $(n+1)$-st roots of unity

Theorem (Dyckerhoff-J-Lekili 2019)

There are equivalences of triangulated A_∞-categories

$$\begin{tikzcd}
\mathrm{perf}(A_{n,d}^{\vee}) \arrow[dr, "\simeq"'] \arrow[rr, leftrightarrow, "\text{Koszul Duality}"] & & \mathrm{perf}(A_{n,d}) \arrow[dl, "\simeq"] \arrow[dd, leftrightarrow, dotted, "\text{[Beckert]}"] & & \\
& \mathcal{W}(\mathrm{Sym}^d(\mathbb{D}), \Lambda_n^{(d)}) & & \mathcal{W}(\mathrm{Sym}^{n-d}(\mathbb{D}), \Lambda_n^{(n-d)}) \arrow[ul, "\simeq"'] & \\
n \geq d \geq 1 & & \mathrm{perf}(A_{n,n-d}) \arrow[ul, "\simeq"'] \arrow[ur, "\simeq"] \arrow[rr, leftrightarrow, "\text{Koszul Duality}"'] & & \mathrm{perf}(A_{n,n-d}^{\vee}) \arrow[ul, "\simeq"']
\end{tikzcd}$$

$A_{n,d}$: ***d-dimensional Auslander algebra*** of type $\mathbb{A}_{n-d+1}$ [Iyama]

Higher Auslander algebras of type $\mathbb{A}$

[Oppermann-Thomas]

$I, J \subset \{1, \dots, n\}$ with $\#I = \#J = d$

Write $I \rightsquigarrow J$ if
$$i_1 \leq j_1 \leq i_2 \leq j_2 \leq \cdots \leq i_d \leq j_d$$

$$A_{n,d} = \bigoplus_{I \rightsquigarrow J} \mathbf{k} \cdot f_{JI}$$

$$f_{KJ} \cdot f_{JI} = \begin{cases} f_{KI} & \text{if } I \rightsquigarrow K \\ 0 & \text{otherwise} \end{cases}$$

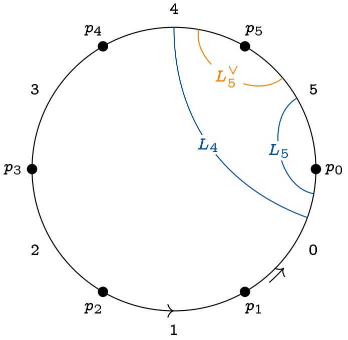
No higher products!

$$A_{n,1} : \mathbf{k}(1 \to 2 \to \cdots \to n)$$

$$A_{n,2} : \begin{tikzcd} 12 \arrow[r] & 13 \arrow[r] \arrow[d] & 14 \arrow[r] \arrow[d] & \cdots \arrow[r] & 1n \arrow[d] \\ & 23 \arrow[r] & 24 \arrow[r] \arrow[d] & \cdots \arrow[r] & 2n \arrow[d] \\ & & 34 \arrow[r] & \cdots \arrow[r] & 3n \arrow[d] \\ & & & \ddots & \vdots \arrow[d] \\ & & & & n-1,n \end{tikzcd}$$

Orlov functors & stop removal functors

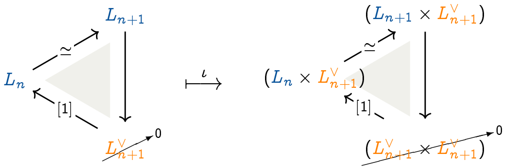
Identify $\Lambda_n = \Lambda_{n+1} \setminus \{p_{n+1}\}$



ι : **Orlov functor**

$$\iota : \mathcal{W}(\mathrm{Sym}^d(\mathbb{D}), \Lambda_n^{(d)}) \hookrightarrow \mathcal{W}(\mathrm{Sym}^{d+1}(\mathbb{D}), \Lambda_{n+1}^{(d+1)})$$

product with small arc $L_{n+1}^\vee$ near stop p_{n+1}



$$\mathcal{W}(\mathrm{Sym}^d(\mathbb{D}), \Lambda_n^{(d)}) \underset{\iota_R}{\overset{\iota_L}{\leftleftarrows\!\!\xrightarrow{\ \iota\ }}} \mathcal{W}(\mathrm{Sym}^{d+1}(\mathbb{D}), \Lambda_{n+1}^{(d+1)}) \underset{s_R}{\overset{s_L}{\leftleftarrows\!\!\xrightarrow{\ s\ }}} \mathcal{W}(\mathrm{Sym}^{d+1}(\mathbb{D}), \Lambda_n^{(d+1)})$$

s : **Stop-removal** functor

Higher Waldhausen $S_\bullet$-construction (Dyckerhoff, Poguntke)

Corollary (DJL, Dyckerhoff-J-Walde)

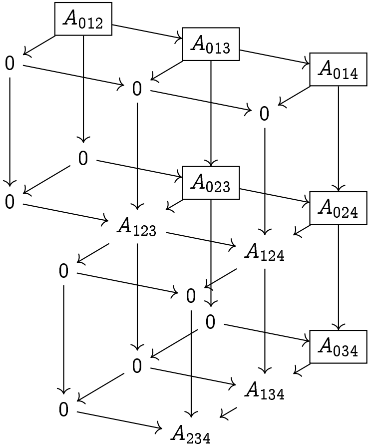
$\mathcal{A} = \operatorname{perf} A$

$$\mathcal{W}(\mathrm{Sym}^d(\mathbb{D}), \Lambda_n^{(d)}) \otimes A \simeq S_n^{(d)}(\mathcal{A})$$

$S_\bullet^{(d)}(\mathcal{A})$: ***d-dim Waldhausen $S_\bullet$-construction***

Theorem (Poguntke)

$$K(\mathcal{A}) \simeq \Omega^d |S_\bullet^{(d)}(\mathcal{A})^{\simeq}|$$



An object of $S_4^{(2)}(\mathcal{A})$
all cubes are *bicartesian*

$$
\begin{tikzcd}[column sep=small]
 & & & & & \mathcal{W}_0^{(0)} \arrow[dr, tail] & & & & & \\
 & & & & \mathcal{W}_1^{(0)} \arrow[ur, two heads] \arrow[dr, tail] & & \mathcal{W}_1^{(1)} \arrow[dr, tail] & & & & \\
 & & & \mathcal{W}_2^{(0)} \arrow[ur, two heads] \arrow[dr, tail] & & \mathcal{W}_2^{(1)} \arrow[ur, two heads] \arrow[dr, tail] & & \mathcal{W}_2^{(2)} \arrow[dr, tail] & & & \\
 & & \mathcal{W}_3^{(0)} \arrow[ur, two heads] \arrow[dr, tail] & & \mathcal{W}_3^{(1)} \arrow[ur, two heads] \arrow[dr, tail] & & \mathcal{W}_3^{(2)} \arrow[ur, two heads] \arrow[dr, tail] & & \mathcal{W}_3^{(3)} \arrow[dr, tail] & & \\
 & \mathcal{W}_4^{(0)} \arrow[ur, two heads] \arrow[dr, tail] & & \mathcal{W}_4^{(1)} \arrow[ur, two heads] \arrow[dr, tail] & & \mathcal{W}_4^{(2)} \arrow[ur, two heads] \arrow[dr, tail] & & \mathcal{W}_4^{(3)} \arrow[ur, two heads] \arrow[dr, tail] & & \mathcal{W}_4^{(4)} \arrow[dr, tail] & \\
\iddots \arrow[ur, two heads] & & \vdots \arrow[ur, two heads] & & \vdots \arrow[ur, two heads] & & \vdots \arrow[ur, two heads] & & \vdots \arrow[ur, two heads] & & \ddots
\end{tikzcd}
$$

$$\mathcal{W}_n^{(d)} = \mathcal{W}(\mathrm{Sym}^d(\mathbb{D}), \Lambda_n^{(d)})$$

$$\operatorname{rank} K_0(\mathcal{W}_n^{(d)}) = \binom{n}{d} = \binom{n}{n-d} = \operatorname{rank} K_0(\mathcal{W}_n^{(n-d)})$$

$$\mathcal{W}_n^{(d-1)} \overset{\iota}{\rightarrowtail} \mathcal{W}_{n+1}^{(d)} \overset{s}{\twoheadrightarrow} \mathcal{W}_n^{(d)}$$

Thank you for your attention!

https://arxiv.org/abs/1911.11719