Stable ∞-categories: Localisations & Recollements

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"Two weeks of Sitting"

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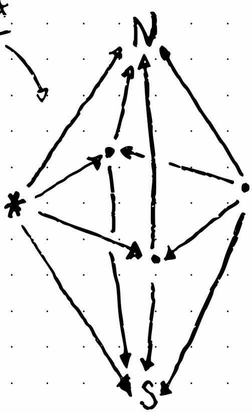
Why higher categories?

$\mathcal{C}$: category & W: a class of morphisms in $\mathcal{C}$

$\mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$ localisation

poset



$\mathcal{C} = W$

in $\mathcal{C}[W^{-1}]$:

$$\begin{tikzcd} & \cdot \arrow[dl] & \\ * \arrow[dr] & & \cdot \arrow[ul] \\ & \cdot \arrow[ur] & \end{tikzcd} \quad \exists_{\gamma}! \doteq id_*$$

moreover

$\mathcal{C}[W^{-1}] \simeq * \circlearrowright id$

$\rightsquigarrow$ Categorical localisations do not see the full homotopy type of a category

We would like

$$\mathcal{C}[W^{-1}] \simeq \Pi_\infty(S^2)$$

! not necessarily a (1-) category

"faithful algebraic avatar" of the 2-dim sphere

Grothendieck's Homotopy Hypothesis

$\forall X$: space $\rightsquigarrow \pi_\infty(X)$: fundamental ∞-gpd

obj: points x of X

1-morph: paths $x \xrightarrow{f} y$ in X

2-morph: homotopies
$$\begin{tikzcd} x \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & y \arrow[from=U, to=D, Rightarrow, "h"] \end{tikzcd}$$

$\vdots$

$(n+1)$-morph: homotopies between n-morph.

$\vdots$

$\pi_\infty(X) \overset{\text{extract}}{\rightsquigarrow} \forall x \in X\ \forall n \geq 0\ \pi_n(X, x)$

$$\boxed{\underline{\text{GHH}} \quad \{\text{spaces}\} \xrightarrow{\sim} \{\infty\text{-groupoids}\} \\ X \longmapsto \pi_\infty(X)}$$

<u>Rmk</u> $\{\text{sets}\} \subset \{\text{spaces}\}$

$X \longmapsto (X, \text{discr.})$

$\mathcal{C}$: $(\infty, 1)$-category

$\forall x, y \in \text{ob}(\mathcal{C}) \rightsquigarrow \text{Map}_{\mathcal{C}}(x, y)$ "space" of maps

+ coherently assoc. composition law

Towards ∞-categories

$\mathcal{C}$: small cat. $\underset{\text{Quillen}}{\rightsquigarrow}$ $B\mathcal{C}$: classifying space of $\mathcal{C}$

$B\mathcal{C}$ is built out of <u>n-simplices</u> in $\mathcal{C}$:

0-simplex $\quad x_0$

1-simplex $\quad x_0 \to x_1$

2-simplex

$$\begin{tikzcd} & x_1 \arrow[dr] & \\ x_0 \arrow[ur] \arrow[rr] & {} \arrow[u, phantom, "="] & x_2 \end{tikzcd}$$

3-simplex

$$\begin{tikzcd} & x_1 \arrow[dr] \arrow[dd] & \\ x_0 \arrow[ur] \arrow[rr] \arrow[dr] & & x_2 \arrow[dl] \\ & x_3 & \end{tikzcd}$$

$[n] = \{0 < 1 < \cdots < n\}$ "universal n-simplex"

$\mathrm{Fun}([n], \mathcal{C})$: set of <u>n-simplices</u> in $\mathcal{C}$

$\Delta := \{[n] \mid n \geq 0\} \subset \mathrm{Cat}$ full subcat

$\uparrow$ <u>simplex category</u>

Functors $\Delta^{op} \to \text{Set}$ are called <u>simplicial sets</u>

<u>Def</u> The <u>nerve</u> of $\mathcal{C}$ is the simplicial set.

$$N(\mathcal{C}) : \Delta^{op} \longrightarrow \text{Set}, \quad [n] \longmapsto \text{Fun}([n], \mathcal{C})$$

$$\boxed{\begin{array}{l} \underline{n\text{-simplex}} \quad \Delta^n := \Delta(-, [n]) : \Delta^{op} \to \text{Set} \\ \qquad\qquad \Delta^n = N([n]) \end{array}}$$

<u>Prop</u> There is an adjunction

$$\tau : \text{Set}_\Delta \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \text{Cat} : N$$

where $\tau(\Delta^n) = [n]$. Moreover

$$N : \text{Cat} \hookrightarrow \text{set}$$

is fully faithful (revisit later)

<u>Def</u> X : space

$$\text{Sing}(X) : \Delta^{op} \to \text{set}, \quad [n] \mapsto \text{Map}(|\Delta^n|, X)$$

where $|\Delta^n| := \{ t \in [0,1]^{n+1} \mid t_0 + t_1 + \cdots + t_n = 1 \}$

Prop There is an adjunction

$$|-| : \mathrm{Set}_\Delta \rightleftarrows \mathrm{Top} : \mathrm{Sing}$$

where $|\Delta^n|$ is as above.

Def The <u>classifying space</u> of $\mathcal{C}$ is

$$B\mathcal{C} := |N(\mathcal{C})|$$

$\gamma : 1_{\mathrm{Set}_\Delta} \longrightarrow \mathrm{Sing}|-|$ unit

$$\rightsquigarrow \quad \mathcal{C} \xrightarrow{\sim} \tau N(\mathcal{C}) \xrightarrow{\tau(\gamma)} \underbrace{\tau\, \mathrm{Sing}(B\mathcal{C})}_{\substack{\cong \\ \mathcal{C}[W^{-1}]}}$$

N is ff

W : <u>all</u> morphisms

$$\rightsquigarrow \quad \begin{tikzcd} N(\mathcal{C}) \arrow[r, "\gamma"] \arrow[dr, "\forall f"'] & \mathrm{Sing}(B\mathcal{C}) \arrow[d, dashed, "\mathrm{Sing}(\bar{f})"] \arrow[dl, phantom, "="] \\ & \mathrm{Sing}(X) \end{tikzcd}$$

$\rightsquigarrow$ What are $N(\mathcal{C})$ & $\mathrm{Sing}(X)$?

Yoneda's Lemma: $X \in Set_\Delta$

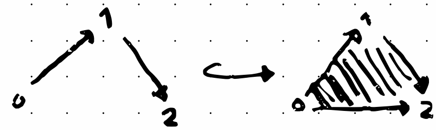
$\forall n \geq 0$: $Hom_{Set_\Delta}(\Delta^n, X) \cong X_n$

$[1] \xrightarrow{!} [0] \xrightarrow{x} \mathcal{C}$ $x \in ob(\mathcal{C})$

$$\begin{tikzcd}
0 \arrow[r, mapsto] \arrow[d] & 0 \arrow[r, mapsto] \arrow[d, "id"] & x \arrow[d, "id_x"] \\
1 \arrow[r, mapsto] & 0 \arrow[r, mapsto] & x
\end{tikzcd}$$

Composition law in $\mathcal{C}$:

$$\begin{tikzcd}
\Delta^{\{1\}} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "PO"] & \Delta^{\{0,1\}} \arrow[d, hook] \arrow[ddr, bend left] & \\
\Delta^{\{1,2\}} \arrow[r, hook] \arrow[drr, bend right] & \Lambda^2_1 \arrow[dr, hook] & \\
& & \Delta^2
\end{tikzcd}$$



$\Lambda^2_1 \hookrightarrow \Delta^2$

$$Hom_{Set_\Delta}(\overset{N([2])}{\overset{\shortparallel}{\Delta^2}}, N(\mathcal{C})) \underset{\text{composition}}{\overset{\sim}{\rightleftarrows}} Hom_{Set_\Delta}(\Lambda^2_1, \mathcal{C})$$

$\cong$ {composable pairs in $\mathcal{C}$ + their composite} $\cong$ {composable pairs in $\mathcal{C}$}

$$\Lambda^n_i := \bigcup_{i \in I \subsetneq [n]} \Delta^I \qquad 0 \leq i \leq n$$

$\boxed{\text{Union of all } n\text{-1 dim faces of } \Delta^n \text{ containing the vertex } i}$

i-th horn in Δ^n

Prop $X \in \mathrm{Set}_\Delta$ is in the ess. image of
$$N : \mathrm{Cat} \hookrightarrow \mathrm{Set}_\Delta$$
if and only if
$\forall n \geq 0 \quad \forall 0 < i < n$ the map
$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, X) \longrightarrow \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_i, X)$$
is <u>bijective</u>

Prop X : space
$\Rightarrow \forall n \geq 0 \quad \forall 0 \leq i \leq n$ the map
$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, X) \longrightarrow \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_i, X)$$
is <u>surjective</u>

Def (Boardman & Vogt)

An <u>∞-category</u> is a simplicial set $\mathcal{C}$ such that $\forall n \geq 0$ $\boxed{\forall 0 < i < n}$ the map
$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathcal{C}) \longrightarrow \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_i, \mathcal{C})$$
is $\boxed{\text{surjective}}$

(Naive) ∞-category theory

..., Joyal, Lurie, ...

$\mathcal{C} : \Delta^{op} \longrightarrow \mathrm{Set}$: ∞-cat $\quad \mathcal{C}_n := \mathcal{C}([n])$

$\mathcal{C}_0$: objects

$\mathcal{C}_1$: morphisms

$\boxed{\text{For intuition about } \mathcal{C}_n \text{ see } \underline{\text{dg}}\ \underline{\text{nerve}}}$

$$\Delta^0 \underset{1}{\overset{0}{\rightrightarrows}} \Delta^1 \xrightarrow{f} \mathcal{C} \quad \rightsquigarrow \quad f_0 \to f_1$$

$\mathcal{C}_2$: triangles in $\mathcal{C}$

$$\begin{tikzcd} \Delta^1 \arrow[r, bend left, "01"] \arrow[r, "02"] \arrow[r, bend right, "12"'] & \Delta^2 \arrow[r, "f"] & \mathcal{C} \end{tikzcd}
\qquad
\begin{tikzcd} & f_1 \arrow[dr, "f_{12}"] & \\ f_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & f & f_2 \end{tikzcd}$$

f exhibits f_{02} as a composite of f_{01} with f_{12}

$\mathcal{C} \rightsquigarrow \tau\mathcal{C} \in \mathrm{Cat}$

Boardman & Vogt

$\tau\mathcal{C} \cong \mathrm{Ho}(\mathcal{C})$: homotopy category of $\mathcal{C}$

$\mathrm{ob}\, \mathrm{Ho}(\mathcal{C}) = \mathcal{C}_0$

$$\mathrm{Ho}(\mathcal{C})(x,y) = \{ x \xrightarrow{f} y \text{ in } \mathcal{C}_1 \} / \sim$$

$$f \sim g \iff \exists \quad \begin{tikzcd} & y \arrow[dr, "e_y"] & \\ x \arrow[ur, "g"] \arrow[rr, "f"'] & & y \end{tikzcd} \text{ in } \mathcal{C}_2 , \qquad \begin{tikzcd} \Delta^1 \arrow[r, "!"] \arrow[dr, "e_y"'] & \Delta^0 \arrow[d, "y"] \arrow[dl, phantom, "="] \\ & \mathcal{C} \end{tikzcd}$$

Let $\Lambda^2_1 \longrightarrow \mathcal{C}$ be a horn in $\mathcal{C}$

which we depict as
$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] & & z \end{tikzcd}$$

Since $\mathcal{C}$ is an ∞-category

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^2 \arrow[ur, dotted, "\exists \text{ filler}"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

Note that if $\exists$ $\begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur, "f"] \arrow[rr, "h'"'] & & z \end{tikzcd}$ then

we have a horn $\Lambda^3_1 \longrightarrow \mathcal{C}$:

$$\begin{tikzcd} & 1 \arrow[dr] \arrow[dd] & \\ 0 \arrow[ur] \arrow[rr] \arrow[dr] & & 2 \arrow[dl] \\ & 3 & \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr, "g"] \arrow[dd, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] \arrow[dr, "h'"'] & & z \arrow[dl, "e_z"] \\ & z & \end{tikzcd}$$

(interior and face opposite to 1 are missing)

The existence of a filler to a 3-spx

implies $\exists$ $\begin{tikzcd} & z \arrow[dr, "e_z"] & \\ x \arrow[ur, "h"] \arrow[rr, "h'"'] & & z \end{tikzcd}$ so $h \sim h'$

Def A <u>functor</u> $F : \mathcal{C} \to \mathcal{D}$ is simply a morphism of simplicial sets.

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) := \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{C} \times \Delta^\bullet, \mathcal{D})$$

$\uparrow$ again ∞-category!

A <u>natural transformation</u> is a morphism $F \xrightarrow{\eta} G$ in the ∞-cat $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathcal{D} \end{tikzcd}$$

If J is a small category, a functor $N(J) \longrightarrow \mathcal{C}$ is sometimes called a <u>homotopy coherent diagram</u>

In contrast, a functor $F : J \longrightarrow \tau(\mathcal{C})$ is a <u>homotopy commutative diagram</u>

Note that $N(F) : N(J) \to N\tau(\mathcal{C})$ <u>need not</u> extend to the ∞-cat $\mathcal{C}$ (along the unit map $\mathcal{C} \to N\tau(\mathcal{C})$)

Thm $\mathcal{C}$: ∞-category

There exists a "canonical" functor

$$\mathrm{Map}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{Spaces}$$

where Spaces = ∞-groupoids.

($X \in \mathrm{Set}_\Delta$ is an ∞-groupoid if the horn filling conditions hold for $0 \leq i \leq n$)

Def Given $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, an adjunction between F and G is an equivalence

$$\varphi : \mathrm{Map}_{\mathcal{C}}(-, G(?)) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{D}}(F(-), ?)$$

in the ∞-cat $\mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{D}, \mathrm{Spaces})$.

Def A functor $F : \mathcal{C} \to \mathcal{D}$ is

(a) fully faithful if $\forall x, y \in \mathcal{C}$,

$$F_{xy} : \mathrm{Map}_{\mathcal{C}}(x,y) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{D}}(Fx, Fy)$$

is an equivalence in Spaces

(b) an equivalence if it is fully faithful

$\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ is dense.

Def An <u>initial object</u> in $\mathcal{C}$ is an object $\emptyset \in \mathcal{C}_0$ such that $\forall\, C \in \mathcal{C}$ there exists an equiv.

$$\mathrm{Map}_{\mathcal{C}}(\emptyset, C) \simeq *$$

in the ∞-cat of spaces

terminal simplicial set

Def A: small ∞-cat $f: A \longrightarrow \mathbb{1} = N([0])$

$\mathcal{C}$ <u>admits (co)limits of shape A</u>

if there exist adjunctions

$$\boxed{f^* = \text{constant-diag functor}} \qquad \begin{tikzcd} \mathrm{Fun}(\mathbb{1}, \mathcal{C}) \arrow[r, "f^*"] & \mathrm{Fun}(A, \mathcal{C}) \arrow[l, bend right, "\mathrm{colim}_A"'] \arrow[l, bend left, "\lim_A"] \end{tikzcd}$$

$\rightsquigarrow$ $\exists$ fully faithful functors

$$\mathrm{Colim}_A : \mathrm{Fun}(A, \mathcal{C}) \hookrightarrow \mathrm{Fun}(A^{\triangleright}, \mathcal{C})$$

$$\mathrm{Lim}_A : \mathrm{Fun}(A, \mathcal{C}) \hookrightarrow \mathrm{Fun}(A^{\triangleleft}, \mathcal{C})$$

$$A^{\triangleright} = A * \{\infty\} \quad \& \quad A^{\triangleleft} = \{-\infty\} * A$$

adjoin a terminal object to A.   adjoin an initial object to A.

Stable ∞-categories (after Lurie)

$\ulcorner = \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet \end{tikzcd}$, $\lrcorner = \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet \end{tikzcd}$, $\square = \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} = \Delta^1 \times \Delta^1$

$$(\ulcorner)^{\triangleright} \cong \square \cong (\lrcorner)^{\triangleleft}$$

$$\begin{tikzcd} \mathrm{Colim}_{\ulcorner} : \mathrm{Fun}(\ulcorner, \mathcal{C}) \arrow[r, hook, "ff"] \arrow[dr, "\simeq"'] & \mathrm{Fun}(\square, \mathcal{C}) \\ & \{\text{pushout squares in } \mathcal{C}\} \arrow[u, phantom, "\cup\!\shortmid"] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Lim}_{\lrcorner} : \mathrm{Fun}(\lrcorner, \mathcal{C}) \arrow[r, hook, "ff"] \arrow[dr, "\simeq"'] & \mathrm{Fun}(\square, \mathcal{C}) \\ & \{\text{pullback squares in } \mathcal{C}\} \arrow[u, phantom, "\cup\!\shortmid"] \end{tikzcd}$$

Def An ∞-category $\mathcal{C}$ is stable if

(a) $\exists\, 0 \in \mathcal{C}_0$ zero object

(b) $\exists\, \mathrm{Colim}_{\ulcorner}$ & $\mathrm{Lim}_{\lrcorner}$ for $\mathcal{C}$

(c) The ess. images of $\mathrm{Colim}_{\ulcorner}$ & $\mathrm{Lim}_{\lrcorner}$ in $\mathrm{Fun}(\square, \mathcal{C})$ coincide

"biCartesian squares"

Prop A : small ∞-cat
$\mathcal{C}$: stable ∞-cat

$\Rightarrow \mathrm{Fun}(A, \mathcal{C})$: stable ∞-cat

Def A functor $F : \mathcal{C} \to \mathcal{D}$ between stable ∞-cat's is *exact* if it preserves zero objects and bicartesian squares.

Prop $\mathcal{C}, \mathcal{D}$: stable ∞-cats

$\Rightarrow \mathrm{Fun}^{ex}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ contains the zero objects & is closed under $\mathrm{Colim}_{\ulcorner}$ & $\mathrm{Lim}_{\lrcorner}$ (for $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$)

(in particular $\mathrm{Fun}^{ex}(\mathcal{C}, \mathcal{D})$ is stable)

Thm $\mathcal{C}$: stable ∞-category

$\Rightarrow \mathrm{Ho}(\mathcal{C})$ is canonically a triang. cat.

References (lecture 2)

Cisinski - Higher categories and homotopical algebra (Ch. 4-6)

Lurie - Higher algebra (Ch. 1)

$\mathcal{C}$: stable ∞-cat $\rightsquigarrow (H_0(\mathcal{C}), \Sigma, T)$

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma x \end{tikzcd} \quad , \quad \begin{tikzcd} \Omega y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & y \end{tikzcd} \rightsquigarrow \Sigma : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$$

$x \simeq \Omega \Sigma x \qquad y \simeq \Sigma \Omega y$

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & \operatorname{cone} f \arrow[r] & \Sigma x \end{tikzcd} \rightsquigarrow x \xrightarrow{f} y \to \operatorname{cone} f \to \Sigma x$$

"standard triangles"

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & y \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\square"] & z \arrow[d] \\ 0 \arrow[r] & \operatorname{cone} f \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{cone}(gf) \arrow[d] \\ & 0 \arrow[r] & \operatorname{cone} g \end{tikzcd} \rightsquigarrow \text{Octahedral axiom}$$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[d] \\ 0 \arrow[r] & y \arrow[r] & 0 \end{tikzcd} \rightsquigarrow z = x \oplus y$$

More on the suspension functor (extra)

$\mathcal{C}$: stable ∞-cat $\quad x, y \in \mathcal{C}_0$

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma x \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(x,y) \arrow[dr, phantom, "\mathrm{PB}"] & \mathrm{Map}_{\mathcal{C}}(0,y) \simeq * \arrow[l] \\ * \simeq \mathrm{Map}_{\mathcal{C}}(0,y) \arrow[u] & \mathrm{Map}_{\mathcal{C}}(\Sigma x, y) \arrow[l] \arrow[u] \end{tikzcd}$$

$$\Rightarrow \quad \mathrm{Map}_{\mathcal{C}}(\Sigma x, y) \xrightarrow{\sim} \Omega \mathrm{Map}_{\mathcal{C}}(x,y)$$

in Spaces = ∞-groupoids

⌜ Compare this with the case of a pre-triang. dg k-category $\mathcal{A}$ where $x[1] \in \mathcal{A}$ satisfies, by definition,

$$\mathcal{A}(x[1], -) \underset{\text{q-iso}}{\xrightarrow{\sim}} \mathcal{A}(x, -)[-1]$$

as left dg $\mathcal{A}$-modules ⌟

∞-categorical localisations

$\mathcal{C}$: ∞-category
$\cup\!\!\mid$
W : "class of morphisms" in $\mathcal{C}$

Def An <u>∞-categorical localisation</u> of $\mathcal{C}$ at W is a functor of ∞-categories

$$\gamma : \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]$$

such that

(a) $\forall \mathcal{D}$: ∞-cat the functor

$$\gamma : \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is fully faithful and

(b) the ess. image of γ^* consists of the functors $\mathcal{C} \to \mathcal{D}$ which map all edges in W to equivalences in $\mathcal{D}$

Rmk ∞-cat localisations always exist (mod. usual set-theoretic caveats)

Exercise $\mathrm{Ho}(\mathcal{C}[W^{-1}]) = \mathrm{Ho}(\mathcal{C})[W^{-1}]$

Prop (..., Cisinski, Szumiło/Kapulkin–Szumiło, ...)

$(\mathcal{E}, \mathcal{S})$: Frobenius exact cat.

$W_{\mathcal{S}}$ = $\mathcal{S}$-homotopy equivalences

$\Longrightarrow$ $\mathcal{E}[W_{\mathcal{S}}^{-1}]$ is a stable ∞-category

Moreover $\mathrm{Ho}(\mathcal{E}[W_{\mathcal{S}}^{-1}]) \simeq \underline{\mathcal{E}}_{\mathcal{S}}$.

as triangulated categories.

Prop (..., Blumberg–Gepner–Tabuada,
Cisinski, Nikolaus–Scholze, ...)

$\mathcal{C}$: stable ∞-cat

$\cup\!|$

$\mathcal{D}$: full subcategory closed under bicartesian squares and containing the zero objects

$W_{\mathcal{D}}$:= morphisms f in $\mathcal{C}$ with $\mathrm{cone}(f) \in \mathcal{D}$

$\Longrightarrow$ $\mathcal{C}/\mathcal{D} := \mathcal{C}[W_{\mathcal{D}}^{-1}]$ is stable

Moreover, $\mathrm{Ho}(\mathcal{C}/\mathcal{D}) = \mathrm{Ho}(\mathcal{C})/\mathrm{Ho}(\mathcal{D})$

Verdier quotient

Recollements of stable ∞-categories

Def (BBDG) A <u>recollement</u> of stable ∞-cat's is a diagram of exact functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "i"] & \mathcal{C} \arrow[l, bend right, "i_L"'] \arrow[l, bend left, "i_R"] \arrow[r, "p"] & \mathcal{B} \arrow[l, bend right, hook, "p_L"'] \arrow[l, bend left, hook, "p_R"] \end{tikzcd} =: \mathcal{R}(\mathcal{B}, \mathcal{A})$$

such that

(a) i, p_L & p_R are fully faithful

(b) $\ker p = \operatorname{Im} i$

(c) $i_L \dashv i \dashv i_R$ & $p_L \dashv p \dashv p_R$

Prop There exist bicartesian squares in the stable ∞-cat $\operatorname{Fun}^{ex}(\mathcal{C}, \mathcal{D})$

$$\begin{tikzcd} p_L p \arrow[r, "\text{counit}"] \arrow[d] \arrow[dr, phantom, "\square"] & id_{\mathcal{C}} \arrow[d, "\text{unit}"] \\ \underline{0} \arrow[r] & i i_L \end{tikzcd} \quad \& \quad \begin{tikzcd} i i_R \arrow[r, "\text{counit}"] \arrow[d] \arrow[dr, phantom, "\square"] & id_{\mathcal{C}} \arrow[d, "\text{unit}"] \\ \underline{0} \arrow[r] & p_R p \end{tikzcd}$$

Gluing stable ∞-categories:

$F : \mathcal{B} \longrightarrow \mathcal{A}$ exact functor

$$\begin{tikzcd}
\mathcal{L}^*(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & \mathrm{Fun}(s \to t, \mathcal{A}) \arrow[d, "t"] \\
\mathcal{B} \arrow[r, "F"'] & \mathrm{Fun}(t, \mathcal{A}) \cong \mathcal{A}
\end{tikzcd}$$

$$\mathcal{L}^*(F) = \{(b, a \xrightarrow{\varphi} Fb) \mid b \in \mathcal{B},\ \varphi \text{ in } \mathcal{A}\}$$

$$\begin{tikzcd}
\mathcal{L}_*(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & \mathrm{Fun}(s \to t, \mathcal{A}) \arrow[d, "s"] \\
\mathcal{B} \arrow[r, "F"'] & \mathrm{Fun}(s, \mathcal{A}) \cong \mathcal{A}
\end{tikzcd}$$

$$\mathcal{L}_*(F) = \{(b, Fb \xrightarrow{\psi} a') \mid b \in \mathcal{B},\ \psi \text{ in } \mathcal{A}\}$$

Lemma $\bar{s} : \mathcal{L}^*(F) \xleftrightarrow{\ \sim\ } \mathcal{L}_*(F) : s^+$

Proof

$$\begin{tikzcd}
 & \mathcal{B} \arrow[rr] \arrow[dd, equal] & & \mathrm{Fun}(y, \mathcal{A}) \arrow[dd, equal] \\
\mathcal{L}^*(F) \arrow[ur] \arrow[rr] \arrow[dd, dashed, shift right, "\bar{s}"'] & & \mathrm{Fun}(x \to y, \mathcal{A}) \arrow[ur, "y"] \arrow[dd, shift right, "\mathrm{cone}"'] & \\
 & \mathcal{B} \arrow[rr] & & \mathrm{Fun}(y, \mathcal{A}) \\
\mathcal{L}_*(F) \arrow[ur] \arrow[rr] \arrow[uu, dashed, shift right, "s^+"'] & & \mathrm{Fun}(y \to z, \mathcal{A}) \arrow[ur, "y"] \arrow[uu, shift right, "\mathrm{cocone}"'] &
\end{tikzcd}$$

Thm (Lurie) The functor

$$\mathrm{Rec}(\mathcal{S}t_\infty) \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{S}t_\infty)$$
$$\mathcal{R}(\mathcal{B}, \mathcal{A}) \longmapsto i_R \circ p_L : \mathcal{B} \longrightarrow \mathcal{A}$$

is an equivalence of ∞-categories.

An inverse is given by $F \mapsto \mathcal{L}_*(F)$

Prop For a recollement $\mathcal{R}(\mathcal{B}, \mathcal{A})$ let

$F := i_R \circ p_L : \mathcal{B} \to \mathcal{A}$ "gluing functor"

TFAE

(a) F admits a right adjoint $G : \mathcal{A} \to \mathcal{B}$

(b) $i_R : \mathcal{C} \longrightarrow \mathcal{A}$ admits a right adjoint

$i_R^! : \mathcal{A} \longrightarrow \mathcal{C}$

In this case,

$$\mathcal{L}^*(G) \xrightarrow{\sim} \mathcal{L}_*(F)$$
$$(a, b \xrightarrow{\varphi} G(a)) \mapsto (b, Fb \xrightarrow{\overline{\varphi}} a)$$

Idea By thm, can assume $\mathcal{C} = \mathcal{L}_*(F)$

(a) $\Rightarrow$ (b)

(b) $\overset{\checkmark}{\Leftarrow}$ (a)

adjunctions compose

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, "\exists i_R^!"] \arrow[drr, bend left, "FG \xrightarrow{\varepsilon} \mathrm{id}_{\mathcal{A}}"] \arrow[ddr, bend right, "G"'] & & \\
& \mathcal{L}_*(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & \mathrm{Fun}(s \to t, \mathcal{A}) \arrow[d, "s"] \\
& \mathcal{B} \arrow[r, "F"'] & \mathrm{Fun}(t, \mathcal{A})
\end{tikzcd}$$

R: ring $\rightsquigarrow \mathcal{D}(R) := \mathrm{Ch}(R)[\mathrm{qiso}^{-1}]$: stable ∞-cat

$\uparrow$ ∞-cat. loc!

<u>Thm</u> (Ladkani for rings)

R, S, E: (dg) rings or "ring spectra"

${}_S M_R \in \mathcal{D}(S^{op} \otimes R)$ such that $\underset{\underset{S \otimes_S M_R}{\shortparallel}}{M_R} \in \mathrm{perf}(R)$

${}_E T_R \in \mathcal{D}(E^{op} \otimes R)$ such that the functor

$$- \otimes_E T : \mathcal{D}(E) \xrightarrow{\sim} \mathcal{D}(R)$$

is an equivalence

$$\Longrightarrow \mathcal{D}\begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \simeq \mathcal{D}\begin{pmatrix} E & \underline{\mathrm{Hom}}_R(M, T_R) \\ 0 & S \end{pmatrix}$$

<u>Proof</u>

$$\begin{tikzcd}
\mathcal{D}(E) \arrow[r, "\underline{\mathrm{Hom}}_R(M, - \otimes_E T)"] \arrow[d, "- \otimes_E T"', "\simeq"] \arrow[dr, phantom, "(*)"] & \mathcal{D}(S) \arrow[d, equal] \\
\mathcal{D}(R) \arrow[r, "\underline{\mathrm{Hom}}_R(M, -)"'] & \mathcal{D}(S)
\end{tikzcd}$$

$M_R \in \mathrm{perf}(R)$

"Eilenberg-Watts"

$$\Rightarrow \underline{\mathrm{Hom}}_R(M, - \otimes_E T) \simeq - \otimes_E \underline{\mathrm{Hom}}_R(M, \underset{E \otimes_E T_R}{T_R})$$

$$\mathcal{D}\begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \overset{(**)}{\simeq} \mathcal{L}_*\left(- \otimes_S M\right)$$

$$\underset{adj}{\simeq} \mathcal{L}^*\left(\underline{\mathrm{Hom}}_R(M,-)\right)$$

$$\underset{S}{\simeq} \mathcal{L}_*\left(\underline{\mathrm{Hom}}_R(M,-)\right)$$

$$\underset{(*)}{\simeq} \mathcal{L}_*\left(- \otimes_E \underline{\mathrm{Hom}}_R(M,T_R)\right)$$

$$\simeq \mathcal{D}\begin{pmatrix} E & \underline{\mathrm{Hom}}_R(M,T_R) \\ 0 & S \end{pmatrix} \qquad \blacksquare$$

Sketch of proof of $(**)$:

$$\begin{tikzcd}
\mathcal{D}(R) \arrow[r, "i"] & \mathcal{L}_*(- \otimes_S M) \arrow[l, bend right, "i_L"'] \arrow[l, bend left, "i_R"] \arrow[r, "p"] & \mathcal{D}(S) \arrow[l, bend right, "p_L"'] \arrow[l, bend left, "p_R"]
\end{tikzcd}$$

$$- \otimes_S M = i_R \circ p_L$$

$\mathcal{L}_*(- \otimes_S M)$ is compactly generated

by $X = i(R) \oplus p_L(S)$

Keller (dg)
Schwede-Shipley $\Longrightarrow \mathcal{L}_*(- \otimes_S M) \simeq \mathcal{D}\begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$

References (lecture 3)

Cisinki - Higher categories and homotopical algebra (Ch. 7)

Dyckerhoff - J - Walde - Generalised BGP reflection functors via the Grothendieck construction

Lurie - Higher algebra (App. A.8)