Companion exercises for the lecture series
STABLE ∞-CATEGORIES: LOCALISATIONS AND RECOLLEMENTS

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Exercise 1 (Nerve and geometric realisation)**.** Let $\mathscr{C}$ be an ordinary category which admits all small colimits. Prove the following statements:

(1) Let $F\colon \mathbf{\Delta} \to \mathscr{C}$ be an arbitrary functor. The functor
$$N_F\colon \mathscr{C} \longrightarrow \mathbf{Set}_{\mathbf{\Delta}}, \qquad c \longmapsto \mathscr{C}(F(-), c)$$
admits a left adjoint.

(2) There is an adjunction
$$\tau\colon \mathbf{Set}_{\mathbf{\Delta}} \rightleftarrows \mathbf{Cat} : N$$
such that $\tau(\Delta^n) = [n]$.

(3) There is an adjunction
$$|-|\colon \mathbf{Set}_{\mathbf{\Delta}} \rightleftarrows \mathbf{Top} : \mathrm{Sing}$$
such that
$$|\Delta^n| = \left\{ \underline{t} \in [0,1]^{n+1} \,\middle|\, t_0 + t_1 + \cdots + t_n = 1 \right\}$$
endowed with the usual topology. Given a topological space X, discuss why is Sing(X) an ∞-category. □

***Exercise** 2 (Lurie's differential graded nerve)**.** For a commutative ring $\Bbbk$, we denote the category (small) differential graded $\Bbbk$-categories and differential graded functors between them by $\mathbf{dgCat}_{\Bbbk}$ (see [Kel06] for definitions but note that we grade chain complexes *homologically*). For $n \geq 0$, let $Q^{(n)}$ be the quiver with vertex set $[n]$ and arrows
$$I\colon i \longrightarrow j$$
for each chain $I = \{ i < k_1 < \cdots < k_\ell < j \}$ with $\ell \geq 0$. We endow $Q^{(n)}$ with a grading by setting
$$\deg(I) = \#(I \setminus \{ i, j \}) = \ell.$$
Let $\Bbbk Q^{(n)}$ be the free (non-negatively) graded $\Bbbk$-category generated by $Q^{(n)}$. Each monotone function $\sigma\colon [m] \to [n]$ induces a functor of graded $\Bbbk$-categories $f_\sigma\colon \Bbbk Q^{(m)} \to \Bbbk Q^{(n)}$ given by
$$f_\sigma(I) = \begin{cases} \sigma(I) & \text{if } \sigma|_I \text{ is injective,} \\ \mathrm{id}_k & \text{if } I = \{ i < j \} \text{ and } \sigma(i) = \sigma(j) = k, \\ 0 & \text{otherwise.} \end{cases}$$
We promote $\Bbbk Q^{(n)}$ to a differential graded $\Bbbk$-category $\Bbbk\Delta^n$ by defining the action of the differential
$$d\colon \Bbbk Q^{(n)}(i,j) \longrightarrow \Bbbk Q^{(n)}(i,j)$$
on a chain $I = \{ i < k_1 < \cdots < k_\ell < j \}$ by the formula
$$d(I) = \sum_{1 \leq m \leq \ell} (-1)^m (I \setminus \{ k_m \} - \{ k_m < \cdots < k_\ell < j \} \circ \{ i < k_1 < \cdots < k_m \}).$$
According to Exercise 1 there is an adjunction
$$\Bbbk\colon \mathbf{Set}_{\mathbf{\Delta}} \rightleftarrows \mathbf{dgCat}_{\Bbbk} : N(-)_{dg}$$

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such that $\Bbbk\Delta^n$ is the differential graded $\Bbbk$-category defined above; moreover, for each small differential graded $\Bbbk$-category $\mathscr{A}$, the simplicial set $\mathrm{N}(\mathscr{A})_{dg}$ is an ∞-category [Lur17, Prop 1.3.1.10]. The functor $\mathrm{N}(-)_{dg}$ is called the **differential graded nerve**.

Let $\mathscr{A}$ be a small differential graded $\Bbbk$-category.

(1) Describe the n-simplices of the ∞-category $\mathrm{N}(\mathscr{A})_{dg}$ for $n = 0, 1, 2, 3$. Discuss why is $\mathrm{N}(\mathscr{A})_{dg}$ an ∞-category.

(2) Prove that
$$\mathrm{Ho}\big(\mathrm{N}(\mathscr{A})_{dg}\big) = \mathrm{H}_0(\mathscr{A}),$$
where $\mathrm{H}_0(\mathscr{A})$ is the category with the same objects as $\mathscr{A}$ and sets of morphisms
$$\mathrm{H}_0(\mathscr{A})(x,y) := \mathrm{H}_0(\mathscr{A}(x,y)).$$
□

Exercise 3. Let
$$\begin{tikzcd} \mathscr{A} \arrow[r, "i"] & \mathscr{C} \arrow[l, bend right, "i_{\mathrm{L}}"'] \arrow[l, bend left, "i_{\mathrm{R}}"] \arrow[r, "p"] & \mathscr{B} \arrow[l, bend right, "p_{\mathrm{L}}"'] \arrow[l, bend left, "p_{\mathrm{R}}"] \end{tikzcd}$$
be a recollement of stable ∞-categories. Prove that there is an equivalence of exact functors
$$(i_{\mathrm{R}} \circ p_{\mathrm{L}}) \simeq \Omega(i_{\mathrm{L}} \circ p_{\mathrm{R}}).$$
Hint: There is a biCartesian square of exact functors
$$\begin{array}{ccc} p_{\mathrm{L}} \circ p & \longrightarrow & \mathrm{id}_{\mathscr{C}} \\ \downarrow & \square & \downarrow \\ 0 & \longrightarrow & i \circ i_{\mathrm{L}} \end{array}$$
where $p_{\mathrm{L}} \circ p \to \mathrm{id}_{\mathscr{C}}$ *and* $\mathrm{id}_{\mathscr{C}} \to i \circ i_{\mathrm{L}}$ *are the counit and the unit of the adjunctions* $p_{\mathrm{L}} \dashv p$ *and* $i_{\mathrm{L}} \dashv i$, *respectively.* □

Exercise 4. Let $\mathrm{F}\colon \mathscr{B} \to \mathscr{A}$ be an exact functor between stable ∞-categories. Sketch the construction of recollements of stable ∞-categories
$$\begin{tikzcd} \mathscr{A} \arrow[r, "i"] & \mathscr{L}^*(\mathrm{F}) \arrow[l, bend right, "i_{\mathrm{L}}"'] \arrow[l, bend left, "i_{\mathrm{R}}"] \arrow[r, "p"] & \mathscr{B} \arrow[l, bend right, "p_{\mathrm{L}}"'] \arrow[l, bend left, "p_{\mathrm{R}}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{A} \arrow[r, "j"] & \mathscr{L}_*(\mathrm{F}) \arrow[l, bend right, "j_{\mathrm{L}}"'] \arrow[l, bend left, "j_{\mathrm{R}}"] \arrow[r, "q"] & \mathscr{B}. \arrow[l, bend right, "q_{\mathrm{L}}"'] \arrow[l, bend left, "q_{\mathrm{R}}"] \end{tikzcd}$$
Hint: Define recollement structures on $\mathrm{Fun}(s \to t, \mathscr{A})$ *and use the defining pullback squares*
$$\begin{array}{ccc} \mathscr{L}^*(\mathrm{F}) & \longrightarrow & \mathscr{B} \\ \downarrow & \mathrm{PB} & \downarrow{\scriptstyle \mathrm{F}} \\ \mathrm{Fun}(s \to t, \mathscr{A}) & \xrightarrow[p=t]{} & \mathrm{Fun}(t, \mathscr{A}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathscr{L}_*(\mathrm{F}) & \longrightarrow & \mathscr{B} \\ \downarrow & \mathrm{PB} & \downarrow{\scriptstyle \mathrm{F}} \\ \mathrm{Fun}(s \to t, \mathscr{A}) & \xrightarrow[q=s]{} & \mathrm{Fun}(s, \mathscr{A}) \end{array}$$
to construct morphisms of recollements
$$\begin{tikzcd} \mathscr{A} \arrow[r, "i"] \arrow[d, "\mathrm{id}_{\mathscr{A}}"'] & \mathscr{L}^*(\mathrm{F}) \arrow[l, bend right, "i_{\mathrm{L}}"'] \arrow[l, bend left, "i_{\mathrm{R}}"] \arrow[r, "p"] \arrow[d] & \mathscr{B} \arrow[l, bend right, "p_{\mathrm{L}}"'] \arrow[l, bend left, "p_{\mathrm{R}}"] \arrow[d, "\mathrm{F}"] \\ \mathscr{A} \arrow[r, "i"] & \mathrm{Fun}(s \to t, \mathscr{A}) \arrow[l, bend right, "i_{\mathrm{L}}"'] \arrow[l, bend left, "i_{\mathrm{R}}"] \arrow[r, "p"] & \mathscr{A} \arrow[l, bend right, "p_{\mathrm{L}}"'] \arrow[l, bend left, "p_{\mathrm{R}}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{A} \arrow[r, "j"] \arrow[d, "\mathrm{id}_{\mathscr{A}}"'] & \mathscr{L}_*(\mathrm{F}) \arrow[l, bend right, "j_{\mathrm{L}}"'] \arrow[l, bend left, "j_{\mathrm{R}}"] \arrow[r, "q"] \arrow[d] & \mathscr{B} \arrow[l, bend right, "q_{\mathrm{L}}"'] \arrow[l, bend left, "q_{\mathrm{R}}"] \arrow[d, "\mathrm{F}"] \\ \mathscr{A} \arrow[r, "j"] & \mathrm{Fun}(s \to t, \mathscr{A}) \arrow[l, bend right, "j_{\mathrm{L}}"'] \arrow[l, bend left, "j_{\mathrm{R}}"] \arrow[r, "q"] & \mathscr{A} \arrow[l, bend right, "q_{\mathrm{L}}"'] \arrow[l, bend left, "q_{\mathrm{R}}"] \end{tikzcd}$$
□

Exercise 5. Let R and S be rings and M an $(\mathrm{S}^{\mathrm{op}} \otimes \mathrm{R})$-bimodule. Let $\mathscr{U}_*(- \otimes_{\mathrm{S}} \mathrm{M})$ be the ordinary category whose objects are the pairs
$$(\mathrm{X}, \varphi) = (\mathrm{X}_{\mathrm{S}}, \varphi\colon \mathrm{X} \otimes_{\mathrm{S}} \mathrm{M}_{\mathrm{R}} \to \mathrm{Y}_{\mathrm{R}})$$
where X is a right S-module and φ is a morphism between right R-modules. A morphism $(\mathrm{X}, \varphi) \to (\mathrm{X}', \varphi')$ in $\mathscr{U}_*(- \otimes_{\mathrm{S}} \mathrm{M})$ consists of morphisms
$$f : \mathrm{X} \to \mathrm{X}' \qquad \text{and} \qquad g : \mathrm{Y} \to \mathrm{Y}',$$

where f is a morphism between right S-modules and g is a morphism between right R-modules, such that the diagram

$$\begin{array}{ccc} X\otimes_S M & \xrightarrow{\varphi} & Y \\ {\scriptstyle f\otimes_S \mathrm{id}_M}\downarrow & & \downarrow{\scriptstyle g} \\ X'\otimes_S M & \xrightarrow{\varphi'} & Y' \end{array}$$

commutes. Prove the following statements:

(1) There is a strict pullback diagram of categories

$$\begin{array}{ccc} \mathscr{U}_*(-\otimes_S M) & \longrightarrow & \mathrm{Fun}(s\to t, \mathrm{Mod}(R)) \\ \downarrow & \mathrm{PB} & \downarrow{\scriptstyle s} \\ \mathrm{Mod}(S) & \xrightarrow[-\otimes_S M]{} & \mathrm{Mod}(R) \end{array}$$

Moreover, the category $\mathscr{U}_*(-\otimes_S M)$ is abelian.

(2) There is a recollement of abelian categories

$$\mathrm{Mod}(R) \underset{\overset{\longleftarrow}{\scriptstyle i_R}}{\overset{\overset{\scriptstyle i_L}{\longleftarrow}}{\xrightarrow{\ i\ }}} \mathscr{U}_*(-\otimes_S M) \underset{\overset{\longleftarrow}{\scriptstyle p_R}}{\overset{\overset{\scriptstyle p_L}{\longleftarrow}}{\xrightarrow{\ p\ }}} \mathrm{Mod}(S)$$

such that $-\otimes_S M = i_R\circ p_L$ (give explicit formulas for all the six functors).

(3) The category $\mathscr{U}_*(-\otimes_S M)$ is equivalent to $\mathrm{Mod}\left(\left(\begin{smallmatrix} S & M \\ 0 & R\end{smallmatrix}\right)\right)$ where

$$\left(\begin{smallmatrix} S & M \\ 0 & R\end{smallmatrix}\right) := \left\{\left(\begin{smallmatrix} s & m \\ 0 & r\end{smallmatrix}\right) \mid r\in R,\ s\in S,\ m\in M\right\}$$

viewed as a ring with respect to the usual matrix operations.

(4) Make the appropriate modifications to define an abelian category $\mathscr{U}^*(\mathrm{Hom}_R(M,-))$ and formulate the analogues of (1), (2) and (3) above. What do you notice? □

References

[Kel06] B. Keller, *On differential graded categories*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190. MR 2275593

[Lur17] J. Lurie, *Higher algebra*, May 2017, Available online at `http://www.math.harvard.edu/~lurie/`.