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d-homological algebra

Higher homological phenomena in algebra and geometry

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Abstract

Recently, in the context of representation theory of finite dimensional algebras, Iyama and his coauthors introduced the so-called d-cluster-tilting subcategories of abelian, exact and triangulated categories. In such categories, by definition, the Ext-groups are trivial in degrees $1, \ldots, d-1$. Notwithstanding, these categories exhibit familiar homological properties with Ext^d playing the role of Ext^1. The aim of 'd-homological algebra' is to develop an abstract framework for the investigation of the intrinsic homological properties of d-cluster-tilting subcategories based on analogues of abelian, exact and triangulated categories.

A motivating example

Throughout, d denotes a fixed positive integer. Let $\mathbb{P}^d$ be the projective d-space over an algebraically closed field. Let

$$\mathcal{V} := \operatorname{add}\{\mathcal{O}(i) \mid i \in \mathbb{Z}\} \subseteq \operatorname{vect}\mathbb{P}^d \subset \operatorname{coh}\mathbb{P}^d$$

be the category of vector bundles over $\mathbb{P}^d$ which are isomorphic to a finite direct sum of shifts of the structure sheaf. Basic properties of sheaf cohomology allow one to show that $\mathrm{Ext}^i(\mathcal{V},\mathcal{V}) = 0$ for all $i \in \{1, \ldots, d-1\}$. As a consequence, every exact sequence

$$0 \to X^0 \to X^1 \to \cdots \to X^d \to X^{d+1} \to 0$$

in $\operatorname{vect}\mathbb{P}^d$ with all terms in $\mathcal{V}$ induces exact sequences

$$\begin{aligned} 0 &\to \mathrm{Hom}(-,X^0) \to \mathrm{Hom}(-,X^1) \to \cdots \to \mathrm{Hom}(-,X^d) \to \mathrm{Hom}(-,X^{d+1}) \to \\ &\to \mathrm{Ext}^d(-,X^0) \to \mathrm{Ext}^d(-,X^1) \to \cdots \to \mathrm{Ext}^d(-,X^d) \to \mathrm{Ext}^d(-,X^{d+1}) \to 0 \end{aligned}$$

and

$$\begin{aligned} 0 &\to \mathrm{Hom}(X^{d+1},-) \to \mathrm{Hom}(X^d,-) \to \cdots \to \mathrm{Hom}(X^1,-) \to \mathrm{Hom}(X^0,-) \to \\ &\to \mathrm{Ext}^d(X^{d+1},-) \to \mathrm{Ext}^d(X^d,-) \to \cdots \to \mathrm{Ext}^d(X^1,-) \to \mathrm{Ext}^d(X^0,-) \to 0. \end{aligned}$$

This is in clear analogy with the long exact sequences induced by short exact sequences in abelian or exact categories. Moreover, the category $\mathcal{V}$ is a **d-cluster-tilting subcategory** of $\operatorname{vect}\mathbb{P}^d$, that is

$$\begin{aligned} \mathcal{V} &= \operatorname{add}\left\{X \in \operatorname{vect}\mathbb{P}^d \,\middle|\, \forall\, i \in \{1,\ldots,d-1\}\ \mathrm{Ext}^i(X,\mathcal{V}) = 0\right\} \\ &= \operatorname{add}\left\{Y \in \operatorname{vect}\mathbb{P}^d \,\middle|\, \forall\, i \in \{1,\ldots,d-1\}\ \mathrm{Ext}^i(\mathcal{V},Y) = 0\right\} \end{aligned}$$

while satisfying an additional 'finiteness property' within $\operatorname{vect}\mathbb{P}^d$ which we shall not make precise here.

This example (due to Iyama), as well as other examples arising in (possibly non-commutative) algebraic geometry, commutative algebra and representation theory of finite dimensional algebras provide motivation for developing an abstract framework for investigating the intrinsic homological properties of d-cluster-tilting subcategories.

d-exact sequences, d-abelian categories and d-exact categories

Oversimplifying, homological algebra can be described as the study of short exact sequences in additive categories. In analogy, d-homological algebra, introduced implicitly by Iyama in [3], corresponds to studying suitable analogues of short exact sequences, called ***d*-exact sequences**; these are complexes of the form

$$0 \to X^0 \to X^1 \to \cdots \to X^d \to X^{d+1} \to 0 \tag{1}$$

satisfying the additional property that the sequences of functors

$$0 \to \mathrm{Hom}(-,X^0) \to \mathrm{Hom}(-,X^1) \to \cdots \to \mathrm{Hom}(-,X^d) \to \mathrm{Hom}(-,X^{d+1})$$

and

$$0 \to \mathrm{Hom}(X^{d+1},-) \to \mathrm{Hom}(X^d,-) \to \cdots \to \mathrm{Hom}(X^1,-) \to \mathrm{Hom}(X^0,-)$$

are exact. It is an easy exercise to show that a d-exact sequence is contractible as a complex if and only if its leftmost morphism is a split monomorphism if and only if its rightmost morphism is a split epimorphism. This, among other properties, justifies the analogy between d-exact sequences and short exact sequences. For further elementary properties of d-exact sequences we refer to [5].

In classical homological algebra, one cannot get very far when investigating short exact sequences in arbitrary additive categories. Thus, one focuses on additive categories satisfying particularly good *properties* with respect to short exact sequences (abelian categories), or additive categories with an additional *structure* (exact categories).

In [5], the classes of d-abelian and d-exact categories are introduced together with their basic properties. Moreover, it is shown that d-cluster-tilting subcategories of abelian and exact categories are d-abelian and d-exact categories respectively. Thus, the category $\mathcal{V}$ defined in the previous section is an example of a d-exact category which can be thought of as a higher analogue of the category of vector bundles over $\mathbb{P}^d$.

Central concepts in the theory of d-abelian and d-exact categories are that of d-pushout and d-pullback diagrams. As their classical versions, they are closely related to the bifunctoriality of the Yoneda Ext^d-groups in d-abelian and d-exact categories.

$(d+2)$-angulated categories

The class of $(d+2)$-angulated categories was introduced by Geiß, Keller and Oppermann in [2]. As their name suggests, these are analogues of classical triangulated categories.

Let $\mathcal{U}$ be an additive category and d a positive integer. A **$(d+2)$-angulation** on $\mathcal{U}$ consists on the structure of an autoequivalence $\Sigma\colon \mathcal{U} \to \mathcal{U}$ and a class of distinguished sequences

$$X^0 \to X^1 \to X^2 \to \cdots X^{d+1} \to \Sigma X^0,$$

called **$(d+2)$-angles**, which satisfy axioms analogous to those of triangulated categories. In particular, every morphism in $\mathcal{U}$ is the leftmost morphism in a $(d+2)$-angle and an analogue of Neeman's 'mapping cone axiom' is satisfied. As a consequence of the axioms, every $(d+2)$-angle as above gives rise to an exact sequence of representable functors

$$\cdots \to (-,\Sigma^{-1}X^{d+1}) \to (-,X^0) \to (-,X^1) \to (-,X^2) \to \cdots \to (-,X^{d+1}) \to (-,\Sigma X^0) \to \cdots$$

In particular, every $(d+2)$-angle is a complex. Geiß, Keller and Oppermann showed that d-cluster-tilting subcategories of triangulated categories which are in addition closed under the d-th power of the suspension functor can be endowed with the structure of a $(d+2)$-angulated category.

Algebraic $(d+2)$-angulated categories are defined in [5] as the stable categories of Frobenius d-exact categories. Examples of non algebraic $(d+2)$-angulated categories which are not algebraic are constructed in [1] for odd values of d extending examples of non algebraic triangulated categories constructed previously by Muro, Schwede and Strickland.

Concrete examples of $(d+2)$-angulated categories arise as follows. Let k be a field and $D := \mathrm{Hom}_k(-,k)$ the usual k-duality. Let A be the path algebra of the quiver $1 \xrightarrow{x} 2 \xrightarrow{x} \cdots \xrightarrow{x} d+1$ bound by the relation $x^2 = 0$. The algebra A is finite dimensional of global dimension d. In particular, the derived Nakayama functor $\nu := - \otimes_A^L DA\colon \mathrm{D^b}(\operatorname{mod} A) \to \mathrm{D^b}(\operatorname{mod} A)$ is a Serre functor on the bounded derived category of finite dimensional (right) A-modules. Iyama showed that the subcategory

$$\mathcal{U}(A) := \operatorname{add}\left\{\nu^i(DA)[-di] \in \mathrm{D^b}(\operatorname{mod} A) \,\middle|\, i \in \mathbb{Z}\right\}$$

is a d-cluster-tilting subcategory of $\mathrm{D^b}(\operatorname{mod} A)$ which is closed under the d-th power of the shift functor; in particular, $\mathcal{U}(A)$ is a $(d+2)$-angulated category. More generally, if A is a d-representation-finite algebra in the sense of [4], then the category $\mathcal{U}(A)$ defined above is a $(d+2)$-angulated category. In fact, these examples are one of the main sources of motivation for introducing $(d+2)$-angulated categories. It is important to mention that the introduction of $(d+2)$-angulated categories predates that of d-abelian categories and d-exact categories.

Current research: derived categories of d-abelian categories and d-exact categories

Using results of Iyama and Geiß, Keller and Oppermann, the examples of $(d+2)$-angulated categories described above can be interpreted as a 'derived category' of a certain d-abelian category: Let A and $\mathcal{U}(A)$ be as in the previous section. Then, $\mathcal{M}(A) := \mathcal{U}(A) \cap \operatorname{mod} A$ is a d-cluster-tilting subcategory of $\operatorname{mod} A$; in particular $\mathcal{M}(A)$ is a d-abelian category. Moreover, there is an equality

$$\mathcal{U}(A) = \operatorname{add}\left\{M[di] \in \mathrm{D^b}(\operatorname{mod} A) \,\middle|\, M \in \mathcal{M}(A) \text{ and } i \in \mathbb{Z}\right\}.$$

Thus, one can think of the $(d+2)$-angulated category $\mathcal{U}(A)$ as a 'derived category' of the 'd-hereditary' d-abelian category $\mathcal{M}(A)$.

An important focus of my current research is the construction of the derived category of suitable d-abelian and d-exact categories. Feel free to contact me if you are interested in the obstructions we have found in attempting to construct these derived categories!

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