# $\tau$-Tilting Reduction

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We fix a field $k = \bar{k}$

and

a finite dimensional $k$-algebra $A$.

# $\tau$-Tilting Modules

## Definition (Iyama-Reiten)

$M$ is *$\tau$-rigid* if

$$\mathrm{Hom}_A(M, \tau M) = 0.$$

- $M$: *$\tau$-tilting module* $:\Leftrightarrow |M| = |A|$

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- $M$: *$\tau$-tilting module* $:\Leftrightarrow |M| = |A|$
- $M$: *support $\tau$-tilting $A$-module* $:\Leftrightarrow M$ : $\tau$-tilting $(A/e)$-module

# Relationship with Tilting Theory I

## Remark

- $\operatorname{Hom}_A(M, \tau M) = 0 \implies \operatorname{Ext}^1_A(M, M) = 0$
- $M$: tilting $\Rightarrow$ $M$: $\tau$-tilting
- $A$: hereditary, then

$$M : \tau\text{-tilting} \Leftrightarrow M : \text{tilting}$$

# Functorially Finite Torsion Classes

## Theorem (Iyama-Reiten)

*There is a bijection*

$$\{\textit{f.f. torsion classes in } \operatorname{mod} A\}$$
$$\updownarrow$$
$$\{\textit{supp. } \tau\textit{-tilting } A\textit{-modules}\}/\sim$$

*given by* $\mathcal{T} \mapsto P(\mathcal{T})$ *with inverse* $M \mapsto \operatorname{Fac} M$.

# Relationship with Tilting Theory II

## Remark

- $M$: support $\tau$-tilting module

$$M \text{ is tilting } \Leftrightarrow DA \in \operatorname{Fac} M$$

In particular, $M$ must be sincere.

# Relationship with Cluster-Tilting Theory

$\mathcal{C}$: triangulated category

- Hom-finite
- Krull-Schmidt
- 2-Calabi-Yau
- has a cluster-tilting object $T$

# Cluster-Tilted Algebras

### Theorem (Adachi-Iyama-Reiten)

$\mathrm{Hom}_{\mathcal{C}}(T,-)$ *induces a bijection*

$$\{\textit{cluster-tilting objects in } \mathcal{C}\}/\sim$$
$$\updownarrow$$
$$\{\textit{supp. } \tau\textit{-tilting } \mathrm{End}_{\mathcal{C}}(T)\textit{-modules}\}/\sim$$

# 2-Calabi-Yau Reduction

## Theorem (Iyama-Yoshino)

- $X$: *rigid* $\qquad {}^{\perp}(X[1]) = \{ Y \in \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(Y, X[1]) = 0 \}$

*There is a bijection*

$$\left\{ \begin{matrix} \textit{cluster-tilting objects } T \in \mathcal{C} \\ \textit{such that } X \in \operatorname{add} T \end{matrix} \right\} / \sim$$

$$\updownarrow$$

$$\left\{ \textit{cluster-tilting objects in } \frac{{}^{\perp}(X[1])}{[X]} \right\} / \sim$$

# Bongartz's Completion

## Proposition (Iyama-Reiten)

- $U$ : $\tau$-rigid $A$-module
- $M$ : support $\tau$-tilting module

Then, $U \in \operatorname{add} M$ if and only if

$$\operatorname{Fac} U \subseteq \operatorname{Fac} M \subseteq {}^{\perp}(\tau U).$$

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Then, $U \in \operatorname{add} M$ if and only if

$$\operatorname{Fac} U \subseteq \operatorname{Fac} M \subseteq {}^{\perp}(\tau U).$$

$T := P({}^{\perp}(\tau U))$ is called the Bongartz's completion of $U$.

# Module Categories from Torsion Pairs

- $U$: $\tau$-rigid A-module

There are f.f. torsion pairs:

$$({}^{\perp}(\tau U), \operatorname{Sub} \tau U)$$

$$(\operatorname{Fac} U, U^{\perp})$$

# Module Categories from Torsion Pairs

- $U$: $\tau$-rigid A-module

There are f.f. torsion pairs:

$$({}^{\perp}(\tau U), \operatorname{Sub} \tau U)$$

$$\nwarrow$$

$${}^{\perp}(\tau U) \cap U^{\perp}$$

$$\searrow$$

$$(\operatorname{Fac} U, U^{\perp})$$

# Module Categories from Torsion Pairs

## Theorem

- $B = \operatorname{End}_A(T)$
- $C = B/e_U$

$\operatorname{Hom}_A(T,-)$ *induces an exact equivalence*

$$\mathbb{F} : {}^{\perp}(\tau U) \cap U^{\perp} \longrightarrow \operatorname{mod} C$$

*with inverse* $\mathbb{G} = - \otimes_B T : \operatorname{mod} C \to {}^{\perp}(\tau U) \cap U^{\perp}$.

# $\tau$-Tilting Reduction

$$
\begin{array}{c}
{}^{\perp}(\tau U) \\
\cup\!| \\
\mathcal{T} \\
\cup\!| \\
\operatorname{Fac} U
\end{array}
$$

# $\tau$-Tilting Reduction

$$
\begin{array}{ccc}
{}^{\perp}(\tau U) & \xrightarrow{M \mapsto \frac{M}{tM}} & {}^{\perp}(\tau U) \cap U^{\perp} \\
\cup\!\shortmid & & \cup\!\shortmid \\
\mathcal{T} & & \mathcal{T} \cap U^{\perp} \\
\cup\!\shortmid & & \cup\!\shortmid \\
\operatorname{Fac} U & & 0
\end{array}
$$

$$0 \to tM \to M \to M/tM \to 0$$

is the canonical sequence of $M$ with respect to $(\operatorname{Fac} U, U^{\perp})$.

# $\tau$-Tilting Reduction

$$\begin{array}{ccccc}
{}^{\perp}(\tau U) & \xrightarrow{M \mapsto \frac{M}{tM}} & {}^{\perp}(\tau U) \cap U^{\perp} & \xrightarrow{\mathbb{F}} & \operatorname{mod} C \\
\cup\!| & & \cup\!| & & \cup\!| \\
\mathcal{T} & & \mathcal{T} \cap U^{\perp} & & \mathbb{F}(\mathcal{T} \cap U^{\perp}) \\
\cup\!| & & \cup\!| & & \cup\!| \\
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\end{array}$$

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# $\tau$-Tilting Reduction

## Theorem

*There is an order-preserving bijection*

$$\left\{\begin{array}{c}\textit{f.f. torsion classes } \mathcal{T} \textit{ in } \operatorname{mod} A \textit{ such that} \\ \operatorname{Fac} U \subseteq \mathcal{T} \subseteq {}^{\perp}(\tau U)\end{array}\right\}$$

$$\updownarrow$$

$$\{\textit{f.f. torsion classes in } \operatorname{mod} C\}$$

*given by* $\mathcal{T} \mapsto \mathbb{F}(\mathcal{T} \cap U^{\perp})$ *with inverse* $\mathcal{X} \mapsto \operatorname{Fac} U * \mathbb{G}\mathcal{X}$.

# $\tau$-Tilting Reduction

## Theorem

*There is an order-preserving bijection*

$$\left\{\begin{matrix} \textit{supp. } \tau\textit{-tilting modules } M \in \operatorname{mod} A \\ \textit{such that } U \in \operatorname{add} M \end{matrix}\right\}$$

$$\updownarrow$$

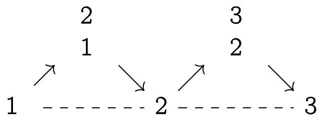
$$\{\textit{supp. } \tau\textit{-tilting modules in } \operatorname{mod} C\}$$

$M \mapsto P(\mathbb{F}(\operatorname{Fac} M \cap U^{\perp}))$ *with inverse* $N \mapsto P(\operatorname{Fac} U * \mathbb{G} \operatorname{Fac} N)$.

# Example

$$A = k(1 \xleftarrow{y} 2 \xleftarrow{x} 3)/xy$$

$\Gamma(\operatorname{mod} A)$ :



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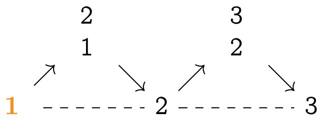


$U = \mathbf{1}$

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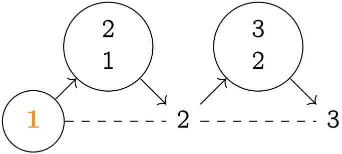


$$U = \mathbf{1} \qquad T = \mathbf{1} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$$

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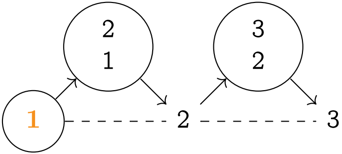


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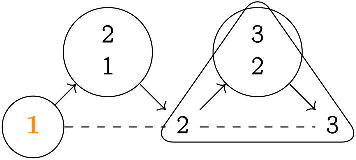


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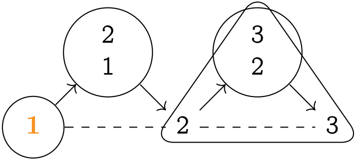


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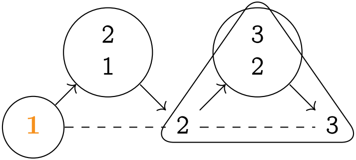


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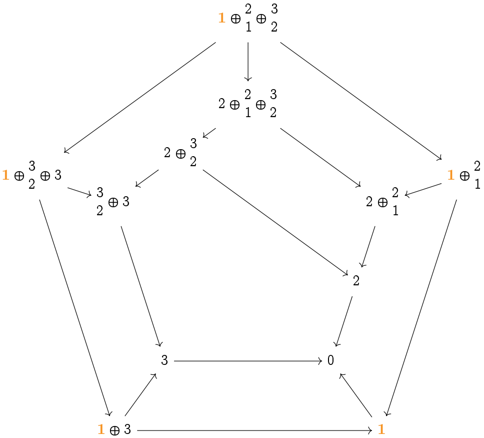

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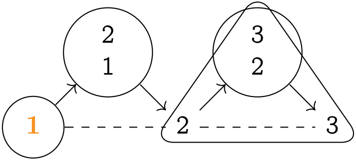


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