REDUCTION OF τ-TILTING MODULES AND TORSION CLASSES

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Throughout this note, A always denotes a finite dimensional associative algebra over a field. We denote by the category of finite dimensional right A-modules by $\operatorname{mod} A$. We denote the Auslander-Reiten translation of an A-module M by τM. We denote the number of pairwise non-isomorphic indecomposable summands of an A-module M by $|M|$. For an A-module M, we denote the smallest full additive subcategory of $\operatorname{mod} A$ which is closed under taking direct summands by $\operatorname{add} M$.

The class of support τ-tilting modules was indtroduced in [1] so as to provide a completion of the class of tilting modules from the point of view of mutation. The definition is as follows:

Definition. [1] Let $M \in \operatorname{mod} A$. We define the following:

(1) We say M is *τ-rigid* if $\operatorname{Hom}_A(M, \tau M) = 0$.
(2) We say that M is *τ-tilting* if M is τ-rigid and $|M| = |A|$.
(3) We say that M is *support τ-tilting* if there exist an idempotent $e \in A$ such that M is a τ-tilting (A/eA)-module.

We denote the set of isomorphism classes of basic support τ-tilting A-modules by $\mathrm{s}\tau\!-\operatorname{tilt} A$.

It is known that support τ-tilting modules have mutation combinatorics similar to mutations of cluster-tilting objects in 2-Calabi-Yau triangulated categories. This phenomenon motivates the study of the exchange graph of $\mathrm{s}\tau\!-\operatorname{tilt} A$ in more detail. For example, it is natural to study the subgraph of the exchange graph induced by all support τ-tilting A-modules which have a given τ-rigid module as a direct summand.

Theorem. [1, 2] *Let U be a τ-rigid A-module. Then the following statements hold:*

(1) *The class*
$$^{\perp}(\tau U) := \{\, M \in \operatorname{mod} A \mid \operatorname{Hom}_A(M, \tau U) = 0 \,\}$$
is closed under factor modules and extensions.

(2) *The class*
$$U^{\perp} := \{\, M \in \operatorname{mod} A \mid \operatorname{Hom}_A(U, M) = 0 \,\}.$$
is closed under submodules and extensions.

(3) *The class $^{\perp}(\tau U)$ admits an* Ext*-projective generator $T = T_U$. Moreover, T is a τ-tilting module.*

(4) *Let $C = \operatorname{End}_A(T)/e_U$, where $e_U \in \operatorname{End}_A(T)$ is the idempotent corresponding to the projective* $\operatorname{End}_A(T)$*-module* $\operatorname{Hom}_A(T, U)$*. Then, the functor* $\operatorname{Hom}_A(T, -) \colon \operatorname{mod} A \to \operatorname{mod} \operatorname{End}_A(T)$ *restricts to an equivalence of categories*
$$F \colon {}^{\perp}(\tau U) \cap U^{\perp} \to \operatorname{mod} C$$

which sends short exact sequences $0 \to L \to M \to N \to 0$ *such that* $L, M, N \in {}^{\perp}(\tau U) \cap U^{\perp}$ *to short exact sequences in* $\operatorname{mod} C$.

(5) *The equivalence* $F : {}^{\perp}(\tau U) \cap U^{\perp} \to \operatorname{mod} C$ *induces a bijection*

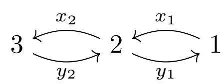
$$\{\, M \in \mathrm{s}\tau\!-\operatorname{tilt} A \mid U \in \operatorname{add} M \,\} \longrightarrow \mathrm{s}\tau\!-\operatorname{tilt} C.$$

Moreover, this bijection is compatible with mutations of support τ*-tilting modules.*

Proof. Statements (1) and (2) are straightforward. The proof of (3) can be found in [1, Thm. 2.9]. The proofs of (4) and (5) are given in [2, Thms. 3.8 and 3.15] respectively. □

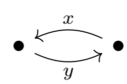
We conlude this note with an example illustrating the Theorem.

Example. Let A be the preprojective algebra of Dynkin type A_3, *i.e.* the algebra given by the quiver

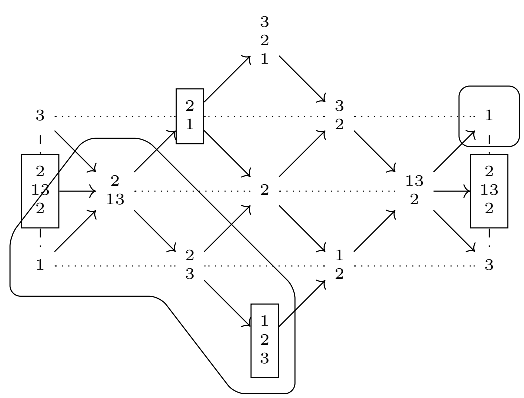


with relations $x_1 y_1 = 0$, $y_2 x_2 = 0$ and $y_1 x_1 = x_2 y_2$.

Let $U = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$, then U is τ-rigid and not a support τ-tilting A-module. The Bongartz completion of U is given by $T = P_1 \oplus P_2 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 13 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$; hence C is isomorphic to the path algebra given by the quiver



with the relations $yx = 0$ and $xy = 0$, *i.e.* the algebra C is isomorphic to the preprojective algebra of Dynkin type A_2. In this case ${}^{\perp}(\tau U)$ consists of all A-modules M such that $\tau U = S_3$ is not a direct summand of $\operatorname{top} M$. On the other hand, it is easy to see that the only indecomposable A-modules which do not belong to $U^{\perp}$ are U, S_2, $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ and P_2. We can visualize this in the Auslander-Reiten quiver of $\operatorname{mod} A$ as follows (note that the dashed edges are to be identified to form a Möbius strip):



The indecomposable summands of T are indicated with rectangles and ${}^{\perp}(\tau U) \cap U^{\perp}$ is encircled. Note that ${}^{\perp}(\tau U) \cap U^{\perp}$ is equivalent to $\operatorname{mod} C$ as shown in the Theorem. We have indicated the embedding of $Q(\mathrm{s}\tau\!-\operatorname{tilt} C)$ in $Q(\mathrm{s}\tau\!-\operatorname{tilt} A)$ in Figure 1 by drawing $Q(\mathrm{s}\tau\!-\operatorname{tilt} C)$ with double arrows.

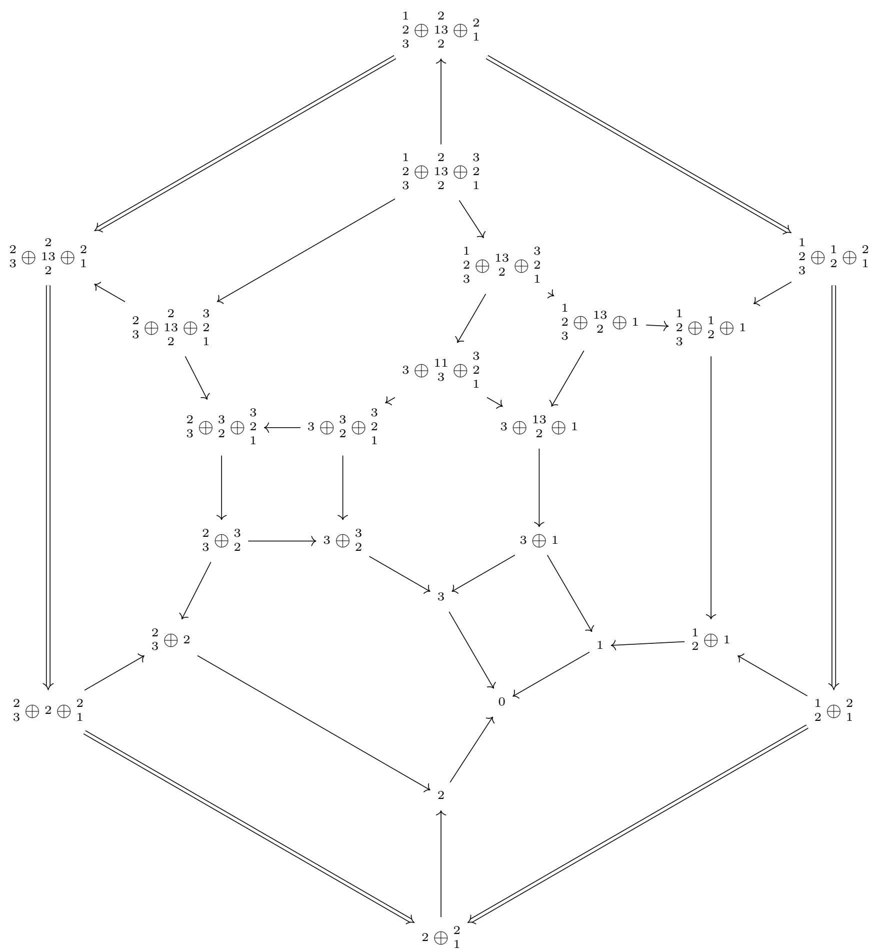


FIGURE 1. Embedding of $s\tau-\operatorname{tilt} C$ in $Q(s\tau-\operatorname{tilt} A)$, see the Example.

REFERENCES

[1] T. Adachi, O. Iyama and I. Reiten: *τ-tilting theory.*. arXiv:1210.1036.

[2] G. Jasso: *Reduction of τ-tilting modules and torsion pairs.*. arXiv:1302.2709.