CLUSTER-TILTED ALGEBRAS OF CANONICAL TYPE AND GRADED QUIVERS WITH POTENTIALS

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Abstract. We describe the effect of mutation on the endomorphism algebras of cluster-tilting objects in the cluster category associated with a canonical algebra in terms of quivers with potentials.

1. Coherent Sheaves over Weighted Projective Lines

Let k be an algebraically closed field. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_t)$ be a tuple of pairwise distinct points of $\mathbb{P}^1_k$ and choose a tuple of positive integers $\mathbf{p} = (p_1, \ldots, p_t)$ such that $p_i \geqslant 2$ for each $i \in \{1, \ldots, t\}$. Following [6], we call the triple $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$ a *weighted projective line*.

Consider the rank one abelian group

$$\mathbb{L} = \mathbb{L}(\mathbf{p}) := \langle \vec{x}_1, \ldots, \vec{x}_t \mid p_1\vec{x}_1 = \cdots = p_t\vec{x}_t =: \vec{c}\rangle$$

and the $\mathbb{L}$-graded commutative algebra

$$S = S(\boldsymbol{\lambda}, \mathbf{p}) = k[x_1, \ldots, x_t]/\langle x_i^{p_i} - \lambda_i' x_2^{p_2} - \lambda_i'' x_1^{p_1} \mid i \in \{3, \ldots, n\}\rangle$$

where $\deg x_i = \vec{x}_i$ and $\lambda_i = [\lambda_i' : \lambda_i''] \in \mathbb{P}^1_k$ for each $i \in \{1, \ldots, t\}$. The category $\operatorname{coh}\mathbb{X}$ of *coherent sheaves over* $\mathbb{X}$ is the quotient of the category $\operatorname{mod}^{\mathbb{L}} S$ by the Serre subcategory of finite length $\mathbb{L}$-graded S-modules. The image of the free module S induces a structure sheaf in $\operatorname{coh}\mathbb{X}$ which we denote, as usual, by $\mathcal{O}$. For a survey of the properties of the category $\operatorname{coh}\mathbb{X}$ we refer the reader to *loc. cit.*. From now on we fix a weighted projective line $\mathbb{X}$ and all sheaves are then assumed to belong to the category $\operatorname{coh}\mathbb{X}$ associated to $\mathbb{X}$ as above. In the following theorem we collect some well-known results that are needed throughout this notes.

Theorem 1.1. [6, 2.2] *The following holds:*

(i) *The category* $\operatorname{coh}\mathbb{X}$ *is an abelian, hereditary, k-linear category with finite dimensional morphism spaces.*

(ii) *Let* $\vec{\omega} := \sum_{i=1}^{t}(\vec{c} - \vec{x}_i) - 2\vec{c}$. *Then there is a bifunctorial isomorphism*

$$\operatorname{Ext}^1_{\mathbb{X}}(X, Y) \cong D\operatorname{Hom}_{\mathbb{X}}(Y, X(\vec{\omega})).$$

Where $X(\vec{\omega})$ *is the sheaf obtained from* X *by shifting the grading by* $\vec{\omega}$.

(iii) *The category* $\operatorname{coh}\mathbb{X}$ *has almost-split sequences induced by the auto-equivalence*

$$\tau : \operatorname{coh}\mathbb{X} \to \operatorname{coh}\mathbb{X}$$

given by $\tau X := X(\vec{\omega})$.

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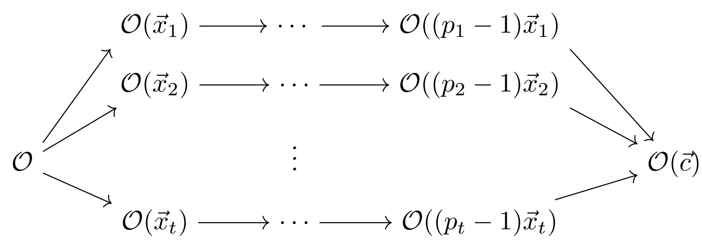
Definition 1.2. A sheaf T is called a *tilting sheaf* if $\mathrm{Ext}^1_{\mathbb{X}}(T,T) = 0$ and the number of pairwise non-isomorphic indecomposable direct summands of T equals $2 + \sum_{i=1}^{t}(p_i - 1)$, the rank of the Grothendieck group of $\operatorname{coh}\mathbb{X}$.

Canonical algebras can be realized as endomorphism algebras of tilting sheaves over weighted projective lines as follows:

Proposition 1.3. [6, Prop. 4.1] *Let T be the following vector bundle:*



Then T is a titling bundle *and* $\mathrm{End}_{\mathbb{X}}(T)$ *is the canonical algebra of parameter sequence* $\boldsymbol{\lambda}$ *and weight sequence* $\mathbf{p}$.

2. The Cluster Category of $\operatorname{coh}\mathbb{X}$

The cluster category associated to a hereditary category was introduced in [4]. Here we recall the basic definitions and necessary results to state our main result.

Definition 2.1. The *cluster category* of $\operatorname{coh}\mathbb{X}$ is the orbit category

$$\mathcal{C} = \mathcal{C}_{\mathbb{X}} := \frac{D^b(\operatorname{coh}\mathbb{X})}{\tau[-1]}$$

whose objects are the objects of $D^b(\operatorname{coh}\mathbb{X})$ and whose morphism spaces are given by

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{X}}(X, (\tau[-1])^i Y)$$

Theorem 2.2. [4, Prop. 1.2][3, Prop. 2.1] *The category $\mathcal{C}$ is a triangulated k-linear category with finite dimensional morphism spaces with the Krull-Schimdt property. Indeed, in this setting, $\mathcal{C}$ is equivalent to the category whose objects coincide with the objects of* $\operatorname{coh}\mathbb{X}$ *and with morphism spaces given by*

$$\mathrm{Hom}_{\mathbb{X}}(X,Y) \oplus \mathrm{Ext}^1_{\mathbb{X}}(X, \tau^{-1}Y)$$

Moreover, $\mathcal{C}$ has the 2-Calabi-Yau property, i.e. *there is a bifunctorial isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \cong \mathrm{Hom}_{\mathcal{C}}(Y, X[2]).$$

Remark 2.3. It follows from Theorem 2.2 that the Hom-spaces in $\mathcal{C}$ have a natural $\mathbb{Z}/2\mathbb{Z}$-grading.

2.1. Cluster-tilting objects and their mutations.

From the point of view of the theory of cluster algebras the so-called cluster-tilting objects play a prominent role.

Definition 2.4. Let $T \in \mathcal{C}$. We say that T is a *cluster-tilting object in $\mathcal{C}$* if $\mathrm{Hom}_{\mathcal{C}}(T, T[1]) = 0$ and the number of pairwise non-isomorphic indecomposable direct summands of T equals $2 + \sum_{i=1}^{t}(p_i - 1)$, the rank of the Grothendieck group of coh $\mathbb{X}$. If T is a cluster-tilting object in $\mathcal{C}$, we call $\mathrm{End}_{\mathcal{C}}(T)$ a *cluster-tilted algebra of canonical type.*

Remark 2.5. A sheaf T, viewed as an object of $\mathcal{C}$, is a cluter-tilting object in $\mathcal{C}$ if and only if it is a tilting sheaf over $\mathbb{X}$, *c.f.* [3, Prop. 2.3].

The class of cluster-tilting objects has an interesting combinatorial structure. It was first studied in [4] in the case of hereditary categories. The general case, of 2-Calabi-Yau triangulated categories, is studied in [8].

Definition 2.6. Let $\mathcal{X}$ be a subcategory of $\mathcal{C}$ and Y an object of $\mathcal{C}$. A morphism $f : X \to Y$ is called a *right $\mathcal{X}$-approximation* if $X \in \mathcal{X}$ and there is an exact sequence of functors

$$\mathrm{Hom}_{\mathcal{C}}(-,X)|_{\mathcal{X}} \xrightarrow{f\cdot} \mathrm{Hom}_{\mathcal{C}}(-,Y)|_{\mathcal{X}} \to 0$$

Left $\mathcal{X}$-approximations are defined dually.

Let $T = T_1 \oplus \cdots \oplus T_n$ be a basic tilting sheaf and fix $k \in \{1,\ldots,n\}$. Then there exist triangles

$$T_k \xrightarrow{f} B \to T'_k \to T_k[1] \quad \text{and} \quad T_k \to B' \xrightarrow{g} T_k \to T'_k[1]$$

where f is a left $\mathrm{add}(\bigoplus_{i\neq k} T_i)$-approximation and g is a right $\mathrm{add}(\bigoplus_{i\neq k} T_i)$-approximation. The *mutation of T in direction k* is by definition

$$T' = \mu_k(T) := T'_k \oplus \bigoplus_{i\neq k} T_i.$$

We note that $T'_k \notin \mathrm{add}\, T$ and that T' is again a basic cluster-tilting object in $\mathcal{C}$.

3. Graded Quivers with Potential

Quivers with potentials were introduced in [5] as a tool to prove certain conjectures about cluster algebras in a broad setting. The graded version, which is the one we present here, was introduced in [2] where the reader may also find other applications of this theory.

3.1. **Jacobian algebras.** Let $Q = (Q_0, Q_1)$ be a finite quiver with no loops or 2-cycles and $d : Q_1 \to \mathbb{Z}/2\mathbb{Z}$ a degree function on the set of arrows of Q. A *potential* for Q is a (possibly infinite) linear combination W of cyclical paths of Q, thus it is an element of the complete path algebra $\widehat{kQ}$. If W is homogeneous with respect to the grading induced by d on $\widehat{kQ}$, in the obvious sense, then we call the triple (Q,W,d) a *graded quiver with potential* (graded QP for short).

Given a graded QP (Q,W,d), there is an associated graded algebra which is the central object of study within this theory, the so-called *Jacobian algebra.* This graded algebra is defined as the quotient

$$\mathrm{Jac}(Q,W,d) := \frac{\widehat{kQ}}{\partial(W)}$$

where $\partial(W)$ is defined as follows: For any cyclical path $a_1 \cdots a_d$ in kQ and an arrow $a \in Q_1$, the *cyclical derivative with respect to a* is defined as

$$\partial_a(a_1 \cdots a_d) = \sum_{a_i = a} a_{i+1} \cdots a_d a_1 \cdots a_{i-1}$$

and then extended by linearity to an arbitray potential. Then $\partial(W)$ is the closure in $\widehat{kQ}$ of the ideal generated by the set $\{\partial_a(W) \mid a \in Q_1\}$.

3.2. **Mutation of graded quivers with potentials.** As with cluster-tilting objects, there is an operation of mutation for quivers with potential. In fact, when considering their graded version there are two different mutations for each direction, the distinction between them ocurring at the level of the grading only. We only consider non-reduced mutations as they are sufficient for the purpouse of this notes.

Let (Q, W, d) be a graded QP with W homogeneous of degree r and $k \in Q_0$. The *left mutation of* (Q, W, d) *in direction* k is the graded QP $\mu_k^L(Q, W, d) = (Q', W', d')$ given as follows:

(i) Q and Q' have the same vertex set.
(ii) All arrows of Q which are not adjacent to k are also arrows of Q' of the same degree.
(iii) Each arrow $a : i \to k$ of Q is replaced in Q' by an arrow $a^* : k \to i$ of degree $d(a) + r$.
(iv) Each arrow $b : k \to j$ of Q is replaced in Q' by an arrow $b^* : j \to k$ of degree $d(b)$.
(v) Each composition $i \xrightarrow{a} k \xrightarrow{b} j$ in Q is replaced in Q' by an arrow $[ba] : i \to j$ of degree $d(a) + d(b)$.
(vi) The new potential is given by

$$W' = [W] + \sum_{i \xrightarrow{a} k \xrightarrow{b} j} [ba]a^*b^*$$

where $[W]$ is the potential obtained from W by replacing each composition $i \xrightarrow{a} k \xrightarrow{b} j$ which appears in W with the corresponding arrow $[ba]$ of Q'.

The *right mutation of* (Q, W, d) *in direction* k, which we denote by $\mu_k^R(Q, W, d)$, is defined in an almost identical manner, only replacing (iii) and (iv) above by the following:

(iii') Each arrow $a : i \to k$ of Q is replaced in Q' by an arrow $a^* : k \to i$ of degree $d(a)$.
(iv') Each arrow $b : k \to j$ of Q is replaced in Q' by an arrow $b^* : j \to k$ of degree $d(b) + r$.

3.3. **Cluster-tilted algebras of canonical type and quivers with potential.** Let T be a basic cluster-tilting object in $\mathcal{C}$ (which is just a tilting sheaf as an object of $\operatorname{coh} \mathbb{X}$). By using results of [9] and [1], we can realize the cluster-tilted algebra $\operatorname{End}_{\mathcal{C}}(T)$ as a Jacobian algebra in the following simple manner.

Let Q_T be the Gabriel quiver of $\Lambda := \operatorname{End}_{\mathbb{X}}(T)$, so that there is an isomorphism

$$\Lambda \cong \frac{kQ_T}{\langle r_1, \dots, r_s \rangle}$$

where $\{r_1, \dots, r_s\}$ are minimal relations. Consider the extended quiver $\tilde{Q}_T$, where for each $k \in \{1, \dots, s\}$ we add an arrow of degree one $r_k^* : j \to i$ if r_i is a relation from the vertex i to the vertex j of Q_T. In words, we extend Q_T by adding an arrow of degree one in the opposite direction of each relation. The arrows of Q_T have degree zero. Then we can form the homogeneous potential of degree one

$$W_T = \sum_{k=1}^{s} r_k r_k^*.$$

Then, by the results of *loc. cit.*, as Λ has global dimension less or equal than 2, there are isomorphisms of graded algebras

$$\mathrm{End}_{\mathcal{C}}(T) \cong \bigoplus_{d \geqslant 0} \mathrm{Ext}^2_{\Lambda}(D\Lambda, \Lambda)^{\otimes d} \cong \mathrm{Jac}(\tilde{Q}_T, W_T, d).$$

Where the algebra in the middle is the 3-preprojective algebra associated with Λ defined by Keller.

4. Main Result

In order to relate the mutation combinatorics of graded quivers with potentials with mutations of cluster-tilting objects in $\mathcal{C}$, we need to group the summands of a given cluster-tilting object $\mathcal{C}$ in two types so as to correspond with the left and right mutations of Section 3.2.

Definition 4.1. [7, Def. 2.9] Let $T = T_1 \oplus \cdots \oplus T_n$ be a tilting sheaf. We say that T_k is an *H-source* if there is an exact sequence

$$0 \to T_k \xrightarrow{\varphi} E \to T'_k \to 0$$

where φ is a left $\mathrm{add}(\bigoplus_{i \neq k} T_k)$-approximation. Analogously, we say that T_k is an *H-sink* if there is an exact sequence

$$0 \to T'_k \to F \xrightarrow{\psi} T_k \to 0$$

where ψ is a right $\mathrm{add}(\bigoplus_{i \neq k} T_k)$-approximation.

We are ready to state the main theorem of this notes, the proof will be provided elsewhere.

Theorem 4.2. *Let $T = T_1 \oplus \cdots \oplus T_n$ be a tilting sheaf. If T_k is an H-source, then there is an isomorphism of graded algebras*

$$\mathrm{End}_{\mathcal{C}}(\mu_k(T)) \cong \mathrm{Jac}(\mu_k^L(\tilde{Q}_T, W_T, d)).$$

Analogously, if T_k is an H-sink, then there is an isomorphism of graded algebras

$$\mathrm{End}_{\mathcal{C}}(\mu_k(T)) \cong \mathrm{Jac}(\mu_k^R(\tilde{Q}_T, W_T, d)).$$

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