Brackets, trees and the Borromean rings

Gustavo Jasso[1]

We describe some of the beautiful mathematical structures that arise from the study of the associativity equation. Our journey will bring us from combinatorics to abstract algebra, with brief excursions through geometry and topology along the way.

1 The combinatorics of bracketings

In this snapshot, we will have a glimpse at the wonderful complexity that emerges from the familiar associativity equation:

$$((ab)c) = (a(bc)).$$

Let us warm up with the following counting problem considered by Catalan [3] in the 1800s:[2]

> Consider a string of $n+1$ letters, say $a_0a_1a_2\cdots a_n$. How many distinct ways are there to insert n pairs of correctly matched parentheses so that these define n binary products on this string?

[1] Gustavo Jasso's research was partially supported by the Swedish Research Council (Vetenskapsrådet) Research Project Grant 2022-03748 'Higher structures in higher-dimensional homological algebra.' The author wishes to thank Bernhard Keller for comments on an earlier version of the snapshot.

[2] This problem is equivalent to that of counting the number of triangulations of a regular convex polygon proposed by Euler in the 1750s.

For a string with one letter there is a single (empty) bracketing a_0, and there is also a single bracketing (a_0a_1) for a string with two letters. For a string with three letters there are exactly two bracketings, $((a_0a_1)a_2)$ and $(a_0(a_1a_2))$. If we now consider the case of a string with four letters we see that there are five distinct bracketings: $(((a_0a_1)a_2)a_3)$, $((a_0(a_1a_2))a_3)$, $((a_0a_1)(a_2a_3))$, $(a_0((a_1a_2)a_3))$ and $(a_0(a_1(a_2a_3)))$.

Let us now tackle the general case. We observe that the first inner pair of parentheses in a bracketing of the string $a_0a_1a_2\cdots a_n$ divides it into a left block $a_0a_1\cdots a_k$ and a right block $a_{k+1}\cdots a_n$. For example, according to the number $k+1$ of letters in the left block (underlined for clarity), the possible bracketings of a string with four letters can be arranged as follows:

$$\begin{array}{lll}
(\underline{a_0}((a_1a_2)a_3)) & (\underline{a_0}(a_1(a_2a_3))) & k=0 \\
(\underline{(a_0a_1)}(a_2a_3)) & & k=1 \\
(\underline{((a_0a_1)a_2)}a_3) & (\underline{(a_0(a_1a_2))}a_3) & k=2.
\end{array}$$

Any bracketing of a string with $k+1$ letters can appear within the left block $a_0a_1\cdots a_k$ and, similarly, any bracketing of a string with $n-k$ letters can appear within the right block $a_{k+1}\cdots a_n$. Thus, if we let C_n be the number of distinct bracketings on a string of $n+1$ letters, we obtain Segner's recursive formula

$$C_0 = 1, \qquad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad n \geq 0.$$

For example, since

$$C_0 = 1, \quad C_1 = 1, \quad C_2 = 2 \quad \text{and} \quad C_3 = 5,$$

we obtain

$$\begin{aligned}
C_4 &= C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0 \\
&= 1\cdot 5 + 1\cdot 2 + 2\cdot 1 + 5\cdot 1 \\
&= 14.
\end{aligned}$$

From Segner's recursive formula one can derive the closed-form expression

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 0,$$

although the derivation is not completely straightforward. The resulting numbers[3]

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862,\ 16796,\ \ldots$$

[3] See the corresponding entry (A000108) in the Online Encyclopedia of Integer Sequences.

are collectively known as the Catalan numbers and they appear remarkably often throughout mathematics. Stanley's book [13] lists 214 families of mathematical objects that are counted by the Catalan numbers, and new families are discovered on a regular basis.[4] Isn't it remarkable that such a fundamental sequence of numbers arises from simple considerations about bracketings?

2 The Tamari lattices

One of the insights of 20th-century mathematics is the importance of recognising and keeping track of the *structures* that are present among the objects that one investigates. In his 1951 doctoral dissertation [15], Tamari observed that the set (=collection) of bracketings of a string with $n+1$ letters has the structure of a partially ordered set. A partial order on a set X consists of a relation $x \preceq y$ that satisfies the following axioms:

Reflexivity For each element $x \in X$, we have $x \preceq x$.

Antisymmetry If $x \preceq y$ and $y \preceq x$, then $x = y$.

Transitivity If $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

Thus, a partial order tells us when an element of the set is 'smaller' than another one with the caveat that two arbitrary elements are not necessarily comparable.[5] For example, we declare that the bracketing $((a_0a_1)a_2)$ is smaller than the bracketing $(a_0(a_1a_2))$ and indicate this relation with an arrow:

$$((a_0a_1)a_2) \longrightarrow (a_0(a_1a_2)).$$

More generally, for a fixed string of letters, we say that one bracketing is smaller than another one if the latter can be obtained from the former by a sequence of rightwards applications of the associativity law. The resulting structures are called the Tamari lattices.[6] Figure 1 depicts the Tamari lattice for a string with four letters, with an arrow pointing from a lower element to a larger one obtained by a single rightwards application of the associativity law. The Tamari lattice for a string with five letters has a three-dimensional character and is depicted in Figure 2 (the outermost pairs of matching parentheses have been omitted to increase readability).

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[4] See for example Rognerud's snapshot [12], were the previous argument is applied to count a family of objects appearing in the theory of quiver representations.

[5] This is quite different from the usual order $\leq$ on the natural numbers, where given two natural numbers a and b we have that $a \leq b$ or $b \leq a$.

[6] A lattice is a special kind of partially ordered set.

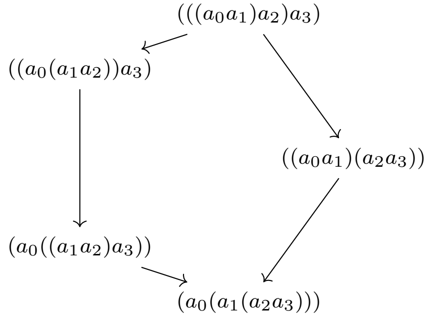


Figure 1: The Tamari lattice for a string with four letters.

We invite the interested reader to peruse the volume [11], published to commemorate the centennial anniversary of Tamari's birth, which contains rather accessible surveys and interesting historical remarks on a wide range of mathematics related to the Tamari lattices (some of them discussed in this snapshot). For us, the illustrations of the Tamari lattices in Figures 1 and 2 suggest a relationship between bracketings and *polytopes*—higher-dimensional analogues of polygons and polyhedra—that we outline in the next section.

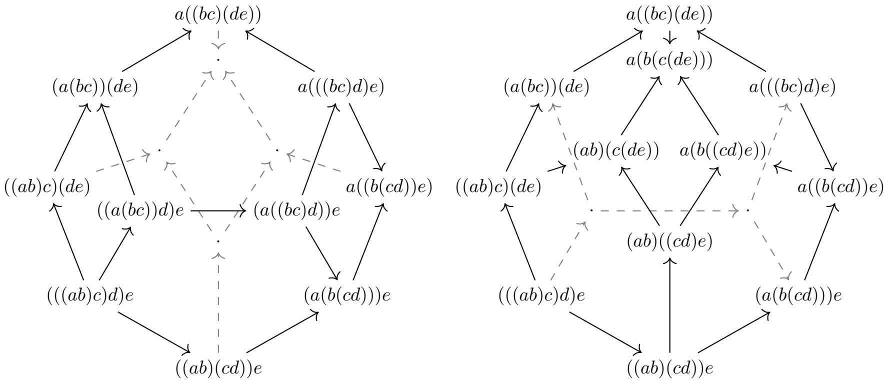
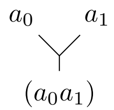


Figure 2: The Tamari lattice for a string with five letters.

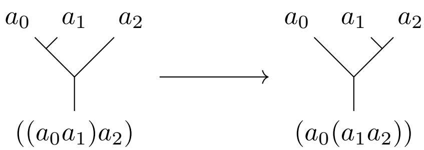
3 From associativity to (mathematical) dendrology

Before continuing our discussion, it is convenient to introduce a way to visualise different bracketings. Namely, each bracketing of a string with $n+1$ letters corresponds to what mathematicians call a 'planar binary rooted tree with $n+1$ leaves.' Rather than giving a precise definition of this mouthful let us look at some examples. The bracketing (a_0a_1) corresponds to the following tree:



(1)

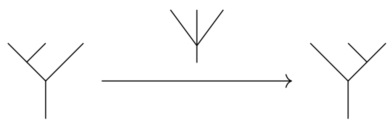
This tree has two leaves at the top and one root pointing to the bottom—it is called 'planar' because it has a fixed embedding in the plane. The two bracketings $((a_0a_1)a_2)$ and $(a_0(a_1a_2))$ correspond to the trees



(2)

which now have three leaves and one root. As before, we have indicated the rightwards application of the associativity law with an arrow. All of these are binary trees since there are exactly two branches emanating upwards from every node.[7] Thus, instead of bracketings, we may consider the Tamari lattices as partial orders on planar binary rooted trees with a fixed number of leaves.

The interpretation of the Tamari lattices in terms of trees reveals further structure that is not immediately apparent from looking at bracketings. Consider the two planar binary rooted trees with three leaves, and notice that they both have a unique internal edge. If we contract this edge to a point we obtain a new planar rooted tree that is no longer binary since three branches/leaves emanate from the unique node. We record the outcome of this procedure pictorially as follows:



[7] Planar binary trees are perhaps more recognisable as genealogical trees, although they are typically drawn upside down in that context.

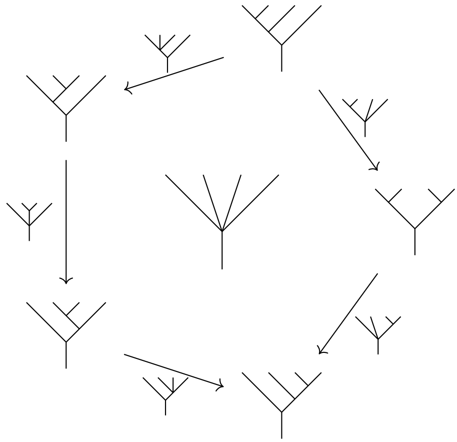


Figure 3: The decorated Tamari lattice for trees with four leaves.

The Tamari lattice for the planar binary rooted trees with four leaves is depicted in Figure 3. Notice that each of the binary trees that is placed on the vertices of the pentagon now has *two* internal edges that we can contract separately into a point. Something conspicuous happens: The new non-binary trees that label the edges of the pentagon each have a single internal edge. If, in any of these trees, we contract this edge into a point we obtain the last planar rooted tree with four leaves. We have included this tree in the Tamari lattice as a further decoration, this time as a label of the interior of the pentagon.

In general, every planar rooted tree can be obtained from the binary ones by contracting one internal edge at a time. Remarkably, the planar rooted trees with $n+1$ leaves label the cells of an $(n-1)$-dimensional polytope that is called an *associahedron* because of its relation to bracketings and the associativity equation.[8] The close relationship between associativity and planar rooted trees can be used, among other things, to establish combinatorial formulas for the inverse with respect to composition of formal power series [9]. As a curious example, the formal power series

$$h(t) = \sum_{n\geq 0} (-1)^{n+1} C_n t^{3n+1},$$

[8] The planar rooted trees with $n+1$ leaves and k internal edges label the $(n-1-k)$-dimensional cells of the corresponding associahedron. It is a nice exercise to label the three-dimensional associahedron in Figure 2 with the planar rooted trees with five leaves.

whose coefficients are the Catalan numbers up to a sign, is its own inverse: $h(h(t)) = t$.

Ceballos and Ziegler [4] remind us that associahedra have been described by Haiman [6] as 'mythical polytope[s]' and, as we shall see below, there is more to the relationship between associahedra and the associativity equation than meets the eye ...

4 Stasheff's A_∞-algebras

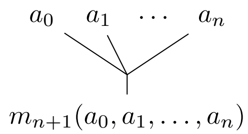
Although associahedra were already present in Tamari's thesis, they were rediscovered in the 1960s by Stasheff—also in his doctoral dissertation [14]. Stasheff's motivation came neither from combinatorics nor from geometry, but from the field of algebraic topology. In a nutshell, algebraic topology studies the qualitative properties of spaces, such as the number of holes in a surface, using tools from abstract algebra. Through his investigations, Stasheff discovered a new class of algebraic structures that he called A_∞-algebras. The precise definition is rather technical so we only sketch some of the main ideas behind it.

An ordinary algebra is a collection of elements that can be added, subtracted and multiplied, but we cannot necessarily divide by non-zero elements and the multiplication does not necessarily satisfy the commutativity law. For example, the polynomials with a fixed number of variables and coefficients in the real numbers form an algebra, and so do the $n \times n$ matrices with real entries. What is important for our discussion is that the multiplication in an algebra is a binary operation: it takes as inputs two elements a and b and outputs their product ab. Thus, we may visualise the multiplication operation in an algebra as a process described by the binary tree in (1). More generally, iterated applications, keeping track of the corresponding bracketing, of the multiplication operation in an algebra can be encoded with a planar binary rooted tree! For example, the two ways to combine the multiplication operations in order to multiply three elements correspond to the trees in (2).

What about the planar rooted trees that are non-binary? Stasheff discovered that some structures that arise in algebraic topology have not only binary operations, but a possibly infinite system of 'higher operations' with $n+1$ inputs and a single output:

$$(a_0, a_1, \ldots, a_n) \longmapsto m_{n+1}(a_0, a_1, \ldots, a_n), \qquad n \geq 0.$$

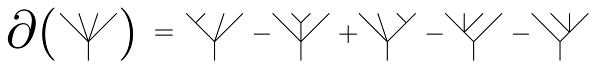
Crucially, the operation m_{n+1} is not obtained by iterated applications of the binary operation $m_2(a_0, a_1) = a_0 a_1$ but is rather a new operation that multiplies $n+1$ elements 'all at once.' With this in mind, it is natural to visualise the operation m_{n+1} as a process described by the tree



Arbitrary planar rooted trees are then obtained by combining the various higher operations. Stasheff also discovered that these operations satisfy an infinite system of equations—the A_∞-equations—that serve as a replacement of the associativity equation:

$$\sum_{r+s+t=n+1} \pm m_{r+1+s}(a_0, a_1, \ldots, a_{r-1}, m_s(a_r, \ldots, a_{s-1}), m_t(a_s, \ldots, a_n)) = 0, \quad n \geq 0.$$

The reader is not expected to parse the A_∞-equations easily (mathematicians struggle to do this the first time they see them as well). We merely wish to highlight the kind of complexity that arises in this context. However, this complexity is not arbitrary: As it turns out, the A_∞-equations are governed by the combinatorial structure of the Stasheff–Tamari associahedra. For example, rewritten in terms of a certain 'boundary operator' $x \mapsto \partial(x)$, the A_∞-equation corresponding to the value $n = 3$ involves precisely the interior and the boundary of the two-dimensional associahedron:[9]



In the above depiction of the A_∞-equation we have replaced combinations of higher operations by the corresponding trees.

Summarising, an A_∞-algebra consists, roughly speaking, of a collection of elements that can be added, subtracted and that are equipped with a system of operations that satisfy the A_∞-equations. One of the many marvels of A_∞-algebras is that they appear not only in algebraic topology but also in fields as diverse as algebraic geometry, representation theory and symplectic geometry. For example, they are crucial to Kontsevich's famous Homological Mirror Symmetry Conjecture that foresees a far-reaching bridge between algebraic and symplectic geometry [8].

Since this is all too abstract, let us look at a concrete example. Consider the polynomials in two variables ε and t with real coefficients, say. We will modify the usual multiplication table for polynomials by declaring that $\varepsilon^2 = 0$.[10] For

[9] The signs that appear are related to the arrows in the Tamari lattice. Do you see how?

[10] For technical reasons that we did not explain we also need to modify the degrees of polynomials by declaring that ε is a variable of degree 1 and that t is a variable of degree 2.

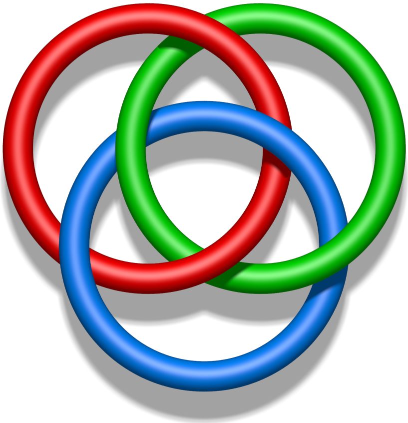


Figure 4: The Borromean rings.

example,

$$(2 + \pi\varepsilon t^2 + t^5)(\varepsilon + 3t^5) = 2\varepsilon + 6t^5 + 3\pi\varepsilon t^7 + \varepsilon t^5 + 3t^{10}$$

since $(\pi\varepsilon t^2)\varepsilon = \pi\varepsilon^2 t^2 = 0$. We now fix a number $n \geq 2$ and introduce the operation

$$m_{n+1}(\underbrace{\varepsilon, \varepsilon, \ldots, \varepsilon}_{n+1 \text{ times}}) = t.$$

This new operation, which cannot be obtained from combinations of the binary multiplication since $\varepsilon^2 = 0$, defines an A_∞-algebra structure on this collection of polynomials and different choices of n give us fundamentally different such structures. Notice also how the new operation allows us to express an algebraic relation of 'higher order' between the variables ε and t, although these are independent from each other if we consider the binary multiplication only.

5 The Borromean rings

Although the definition of an A_∞-algebra is quite technical and somewhat obscure, the higher operations sometimes have interesting interpretations. Here is one such example.

The Borromean rings,[11] depicted in Figure 4, are a configuration of three interlinked circles in three-dimensional space. The distinctive feature of this

[11] Named after the House of Borromeo who included them in their coat of arms.

configuration is that, while the three circles are linked together, if we remove any one circle the remaining two are no longer linked.[12] Using sophisticated machinery from algebraic topology one constructs an A_∞-algebra from the Borromean rings; this A_∞-algebra has three distinguished elements α, β and γ (corresponding roughly to the three circles in the configuration) whose multiplication table looks as follows:

$$\alpha\beta = \beta\alpha = 0, \quad \alpha\gamma = \gamma\alpha = 0 \quad \text{and} \quad \beta\gamma = \gamma\beta = 0.$$

These equations reflect the fact that any two circles in the Borromean rings are unlinked. On the other hand, Massey [10] discovered (in a slightly different language) that there is a non-zero ternary operation

$$m_3(\alpha, \beta, \gamma) \neq 0$$

that witnesses the fact that the circles in the Borromean rings are triply linked!

6 A_∞-algebras in contemporary mathematical research

Even though A_∞-algebras were invented about sixty years ago, many of their properties remain mysterious and their careful study has led to unexpected applications. For example, in their work in (complex) three-dimensional algebraic geometry [5], Donovan and Wemyss formulated a deep conjecture that, thanks to the work of several mathematicians, relates a beautiful family of geometric objects called compound Du Val singularities[13] to a particular class of A_∞-algebras. The recent solution [7] to the conjecture involves a delicate analysis of the qualitative properties of these A_∞-algebras—an approach that was not at all obvious from the original formulation of the conjecture!

[12] An elementary proof of the fact that the Borromean rings are indeed linked can be found in the book [1] by Aigner and Ziegler, a book that we wholeheartedly recommend to the reader.

[13] See also the snapshot by Buchweitz and Faber [2] where the (two-dimensional) Du Val singularities make an appearance.

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