Universal Massey Products in Representation Theory of Algebras

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(joint work with Fernando Muro)

We work over an arbitrary field. Recall Kadeishvili's Intrinsic Formality Criterion [Kad88]:

Theorem. *Let A be a graded algebra whose Hochschild cohomology vanishes in the following bidegrees:*

$$\mathrm{HH}^{p+2,-p}(A,A)=0, \qquad p\geq 1.$$

Then, every minimal A_∞-algebra structure on A is gauge A_∞-isomorphic to the trivial A_∞-structure, whose higher operations $m_{p+2}=0$, $p\geq 1$, vanish.

In our joint work we generalise Kadeishvili's Criterion as follows.

Definition. Fix an integer $d\geq 1$. A graded algebra is *d-sparse* if it is concentrated in degrees that are multiples of d (hence this condition is empty if $d=1$). A *d-sparse Massey algebra* is a pair (A,m) consisting of a d-sparse graded algebra A and a Hochschild cohomology class

$$m\in \mathrm{HH}^{d+2,-d}(A,A), \qquad \mathrm{Sq}(m)=0,$$

of bidegree $(d+2,-d)$ whose Gerstenhaber square vanishes.

For example, if

$$(A, m_{d+2}, m_{2d+2}, m_{3d+2},\dots)$$

is a minimal A_∞-algebra structure on a d-sparse graded algebra A (in which case $m_{i+2}=0$, $i\notin d\mathbb{Z}$, for degree reasons), then $m_{d+2}\in \mathrm{C}^{d+2,-d}(A,A)$ is a Hochschild cocycle whose associated Hochschild cohomology class

$$\{m_{d+2}\}\in \mathrm{HH}^{d+2,-d},$$

its *universal Massey product (of length $d+2$)*, has vanishing Gerstenhaber square

$$\mathrm{Sq}(\{m_{d+2}\})=0.$$

Consequently, the pair $(A,\{m_{d+2}\})$ is a d-sparse Massey algebra.

Remark. It is an easy consequence of the d-sparsity assumption that the universal Massey product of a minimal A_∞-algebra is invariant under A_∞-isomorphisms.

Remark. Universal Massey products of length 3 have been investigated previously in representation theory, see for example [BKS04].

Definition. The *Hochschild–Massey cohomology* of a d-sparse Massey algebra (A,m) is the cohomology

$$\mathrm{HH}^{\bullet,*}(A,m)$$

of the *Hochschild–Massey (cochain) complex*, that is the bigraded cochain complex with components

$$\mathrm{HH}^{p+2,*}(A,A), \qquad p\geq 0,$$

and differential

$$\mathrm{HH}^{\bullet,*}(A,A) \longrightarrow \mathrm{HH}^{\bullet+d+1,*-d}(A,A), \qquad x \longmapsto [m,x],$$

in source bidegrees different from $(d+1,-d)$, where the differential is instead given by the formula by

$$\mathrm{HH}^{d+1,-d}(A,A) \longrightarrow \mathrm{HH}^{2(d+1),-2d}(A,A), \qquad x \longmapsto [m,x]+x^2.$$

Remark. That the differential of the Hochschild–Massey complex squares to zero is a consequence of the Gerstenhaber relations and the assumption $\mathrm{Sq}(m)=0$.

Theorem ([JKM22, Theorem B])**.** *Let (A,m) be a d-sparse Massey algebra whose Hochschild–Massey cohomology vanishes in the following bidegrees:*

$$\mathrm{HH}^{p+2,-p}(A,m)=0, \qquad p>d.$$

Then, any two minimal A_∞-algebras

$$(A, m_{d+2}, m_{2d+2}, m_{3d+2}, \dots) \qquad \text{and} \qquad (A, m'_{d+2}, m'_{2d+2}, m'_{3d+2}, \dots)$$

such that $\{m_{d+2}\} = m = \{m'_{d+2}\}$ are gauge A_∞-isomorphic.

Remark. Kadeishvili's Intrinsic Formality Criterion is indeed a corollary of the above theorem: Take $d=1$ and notice that the hypothesis in the criterion implies that every minimal A_∞-algebra structure on A has vanishing universal Massey product $\{m_3\}=0$.

The proof of the theorem relies in an essential way on an enhanced A_∞-obstruction theory developed by F. Muro in [Mur20a]. We also mention that the theorem is one of the key ingredients in the proof of the main theorem in [JKM22] which, as explained by B. Keller in the Appendix to *loc. cit.*, in a special case yields the final step in the proof of the Donovan–Wemyss Conjecture in the context of the Homological Minimal Model Program for threefolds [DW16, Wem23].

The aforementioned applications of the theorem rely on the following observation: The Hochschild–Massey cochain is equipped with a canonical bidegree $(d+2,-d)$ endomorphism given by

$$\mathrm{HH}^{\bullet,*}(A,A) \longrightarrow \mathrm{HH}^{\bullet+d+2,*-d}(A,A), \qquad x \longmapsto m \smile x,$$

in source bidegrees different from $(d+1,-d)$, where it is given by

$$\mathrm{HH}^{d+1,-d}(A,A) \longrightarrow \mathrm{HH}^{2(d+1)+1,-2d}(A,A), \qquad m \smile x + \{\delta_{/d}\} \smile x^2.$$

Here,

$$\delta_{/d} \in \mathrm{C}^{1,0}(A,A), \qquad x \longmapsto \tfrac{|x|}{d}x,$$

is the fractional Euler derivation (notice that $\frac{|x|}{d}$ is an integer due to the assumption that the graded algebra A is d-sparse). The above endomorphisms is in fact null-homotopic. An explicit bidegree $(1,0)$ null-homotopy is given by

$$\mathrm{HH}^{\bullet,*}(A,A) \longrightarrow \mathrm{HH}^{\bullet+1,*}(A,A), \qquad x \longmapsto \{\delta_{/d}\} \smile x.$$

Thus, a sufficient condition for the assumptions in the theorem to be satisfied is that the components of above endomorphism of the Hochschild–Massey complex of (A, m) are bijective in all non-trivial source bidegrees. The latter condition is satisfied by the d-sparse Massey algebras investigated in [JKM22].

References

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