Derived Endomorphism Algebras in Higher Auslander–Reiten Theory

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(joint work with Fernando Muro)

Let k be a field and A a finite-dimensional algebra over k. Suppose that A is of finite representation type, that is the category $\mathrm{mod}(A)$ of finite-dimensional (right) A-modules admits an additive generator M, say. The algebra $\Gamma := \mathrm{End}_A(M)$ of endomorphisms of M is then an *Auslander algebra*, that is Γ has global dimension at most 2 and dominant dimension at least 2 [Aus71]. The basic paradigm of Auslander–Reiten Theory is that the minimal projective resolutions of simple Γ-modules of projective dimension 2 (the largest possible) correspond to almost-split sequences in $\mathrm{mod}(A)$ [AR75]. More generally, if $d \geq 1$ and M is a d-cluster tilting A-module, then Γ is a $(d+1)$-*dimensional Auslander algebra* in the sense of Iyama [Iya07], that is Γ has global dimension at most $d+1$ and dominant dimension at least $d+1$. We remind the reader that M is a *d-cluster tilting A-module* if the following conditions are equivalent for an indecomposable A-module X:

- X is a direct summand of M.
- For all $0 < i < d$, $\mathrm{Ext}^i_A(X, M) = 0$.
- For all $0 < i < d$, $\mathrm{Ext}^i_A(M, X) = 0$.

Thus, a 1-cluster tilting A-module is simply an additive generator of $\mathrm{mod}(A)$ for the latter two conditions are empty in this case. In this more general context, minimal projective resolutions of simple Γ-modules of projective dimension $d+1$ correspond to *d-almost-split sequences* in $\mathrm{add}(M) \subseteq \mathrm{mod}(A)$, the additive closure of M in $\mathrm{mod}(A)$. Furthermore, up to Morita equivalence, the association $(A, M) \mapsto \mathrm{End}_A(M)$ induces a bijection between:

(1) Pairs (A, M) consisting of a finite-dimensional algebra A and a d-cluster tilting A-module M.
(2) $(d+1)$-Auslander algebras Γ.

The above bijective correspondence is known as the *Auslander–Iyama Correspondence* [Aus71, Iya07].

Suppose now that Λ is a finite-dimensional selfinjective algebra; for simplicity, assume Λ to be basic. We wish to interpret the minimal projective resultions of simple Γ-modules of infinite (!) projective dimension in higher Auslander–Reiten-theoretic terms. For this, it is necessary to enforce a certain periodicity on these resultions. More precisely, we assume that there exists an exact sequence of Λ-bimodules

$$0 \to \Lambda_\sigma \to P_{d+1} \to P_d \to \cdots \to P_2 \to P_1 \to P_0 \to \Lambda \to 0$$

with projective middle terms, where σ is an algebra automorphism of Λ; in this case we say that Λ is *twisted $(d+2)$-periodic with respect to σ*. Let S be a simple Λ-module of infinite projective dimension; applying the tensor product functor $S \otimes_\Lambda -$ to the above exact sequence yields the first part of a projective resolution of S that is 'twisted periodic' since the $(d+2)$-syzygy of S is again a simple Λ-module.

Thus, the minimal total projective resolution of S is completely determined by the automorphism σ and the truncation

$$Q_{d+1} \to Q_d \to \cdots \to Q_2 \to Q_1 \to Q_0 \to \nu Q_0,$$

where Q_0 is the projective cover of S and νQ_0 is its injective hull. It is natural to wish to interpret the latter complex as an almost split $(d+2)$-angle [IY08, GKO13]. Indeed, a theorem of Amiot [Ami07] in the case $d = 1$ and a generalisation by Lin [Lin19] show that the pair $(\mathrm{proj}(\Lambda), - \otimes_\Lambda \Lambda_{\sigma^{-1}})$ admits a $(d+2)$-angulation, where $\mathrm{proj}(\Lambda)$ is the category of finite-dimensional projective Λ-modules. Conversely, if $\mathrm{proj}(\Lambda)$ admits a $(d+2)$-angulated structure, then Λ must be twisted $(d+2)$-periodic with respect to some algebra automorphism [GSS03, GKO13, Han20]. Furthermore, if Λ arises as the endomorphism algebra of a $d\mathbb{Z}$-cluster tilting object in a triangulated category[1] with finite-dimensional morphism spaces, then $\mathrm{proj}(\Lambda)$ admits a $(d+2)$-angulated structure [GKO13]. The main result in [JM22] refines the above to the following more precise statement (the case $d = 1$ was established in [Mur22]):

Theorem (Derived Auslander–Iyama Correspondence)**.** *Let k be a perfect field. There is a bijective correspondence between the following:*

(1) *Quasi-isomorphism classes of DG algebras A such that $H^0(A)$ is a basic finite-dimensional algebra and A is a $d\mathbb{Z}$-cluster tilting object of its perfect derived category $D^c(A)$.*

(2) *Equivalence classes of pairs (Λ, σ) consisting of a basic finite-dimensional algebra Λ and σ is an algebra automorphism such that Λ is twisted* $(d+2)$*-periodic with respect to σ.*

The correspondence is given by $A \mapsto (H^0(A), \sigma)$, where σ is a choice of algebra automorphism of $H^0(A)$ such that $H^{-d}(A) \cong H^0(A)_\sigma$ as $H^0(A)$-bimodules.

The key ingredient in the proof of the theorem is the *restricted universal Massey product (rUMP) of length* $d+2$ associated to any minimal A_∞-model of A [Kad82, Kel01, LH]. By definition, the rUMP of A is the Hochschild cohomology class

$$u_A \in HH^{d+2,-d}(H^0(A), H^*(A))$$

that is the image of the class $\{m_{d+2}\} \in HH^{d+2,-d}(H^*(A), H^*(A))$ of the higher operation $m_{d+2}\colon H^*(A)^{\otimes d+2} \to H^*(A)[-d]$ under the canonical map

$$HH^{d+2,-d}(H^*(A), H^*(A)) \longrightarrow HH^{d+2,-d}(H^0(A), H^*(A)).$$

Indeed, a further main result in [JM22] is the following variant of the above theorem:

Theorem. *Let k be a perfect field. There is a bijective correspondence between the following:*

(1) *Quasi-isomorphism classes of DG algebras A such that $H^0(A)$ is a basic finite-dimensional algebra and A is a $d\mathbb{Z}$-cluster tilting object of its perfect derived category $D^c(A)$.*

[1] That is a (basic) d-cluster tilting object that is isomorphic to its d-fold shift.

(2) *A_∞-isomorphism classes of minimal A_∞-algebras B with the following properties:*
   - *The ordinary algebra B^0 is a basic Frobenius algebra.*
   - *The underlying graded algebra of B is concentrated in degrees that are multiples of d, and there exists an invertible element $\varphi \in B^d$.*
   - *The rUMP $u_B \in HH^{d+2,-d}(B^0, B)$ is invertible in the Hochschild–Tate cohomology (bigraded) algebra $\underline{HH}^{\bullet,*}(B^0, B)$.*

The correspondence associates to a DG algebra A any of its minimal A_∞-models.

It is interesting to investigate in more detail the existence of additional structures on the DG algebras that arise from the Derived Auslander–Iyama Correspondence.

Conjecture. *Let Λ be a basic finite-dimensional algebra that is twisted $(d+2)$-periodic with respect to the Nakayma automorphism ν of Λ. Let A be any DG algebra that corresponds to (Λ, ν) under the Derived Auslander–Iyama Correspondence. Then, A admits a right d-Calabi–Yau structure in the sense of* [KS06].

The conjecture is motivated by the existence of a right d-Calabi–Yau structure on the Amiot–Guo–Keller cluster category [Ami09, Guo11, Kel05a] associated to the derived $(d+1)$-preprojective algebra [Kel11, IO13] of a d-representation finite algebra [IO11], see [KL23] for an announcement of the proof of a much more general theorem on Calabi–Yau structures on Drinfeld quotients.

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