The symplectic geometry of higher Auslander algebras, an overview

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(joint work with Tobias Dyckerhoff, Yankı Lekili)

Let $\mathbf{k}$ be a commutative ring. Let $\mathbb{D}$ be the 2-dimensional unit disk and $\Lambda_n \subset \partial\mathbb{D}$ the set of $(n+1)$-th roots of unity, where $n \geq 0$. To these data one can associate [Aur10b, Aur10a] a *partially wrapped Fukaya category* $\mathcal{W}(\mathbb{D}, \Lambda_n)$, which is an idempotent-complete triangulated A_∞-category. After choosing appropriate generators, the aforementioned Fukaya category can be described combinatorially as the perfect derived category of $\mathbf{k}$-linear representations of the linearly oriented quiver

$$A_n := (1 \to 2 \to \cdots \to n)$$

of Dynkin type $\mathbb{A}_n$. As originally observed by Waldhausen [Wal85] (in a slightly different language), the derived functors induced by the morphisms between the various quivers A_n, $n \geq 0$ are part of a *co*-simplicial object $\mathsf{perf} A_\bullet$. Consequently, for each A_∞-category $\mathcal{A}$ there is an associated *simplicial* object

$$\mathsf{Fun}_{\mathbf{k}}(\mathcal{W}(\mathbb{D}, \Lambda_\bullet), \mathcal{A}) \overset{(a)}{\simeq} \mathsf{Fun}_{\mathbf{k}}(\mathsf{perf} A_\bullet, \mathcal{A}) \overset{(b)}{\simeq} \mathsf{S}(\mathcal{A})_\bullet$$

whose triangulated A_∞-category of n-cells is given by the A_∞-category of A_∞-functors $\mathcal{W}(\mathbb{D}, \Lambda_n) \to \mathcal{A}$. The simplicial object $\mathsf{S}(\mathcal{A})_\bullet$, called the *Waldhausen* S*-construction of* $\mathcal{A}$, is the main ingredient in the construction of the *Waldhausen* K*-theory space* $K(\mathcal{A})$ of $\mathcal{A}$, for we have the formula

$$K(A) := \Omega|\mathsf{S}(\mathcal{A})_\bullet^{\simeq}|.$$

In summary, the quasi-equivalent simplicial objects above provide an explicit connection between

- the (partially) wrapped Floer theory of the 2-dimensional unit disk,
- the derived representation theory of Dynkin quivers of type $\mathbb{A}$ and
- the Waldhausen K-theory of A_∞-categories.

Let $d \geq 1$ be a natural number. In previous work with Dyckerhoff and Walde [DJW19] we have described a higher-dimensional generalisation of the quasi-equivalence (b) above, which now takes the form

$$\mathsf{Fun}_{\mathbf{k}}(\mathsf{perf} A_{\bullet,d}, \mathcal{A}) \simeq \mathsf{S}^{\langle d\rangle}(\mathcal{A})_\bullet \tag{1}$$

and relates the *d-dimensional Waldhausen* S*-construction* $\mathsf{S}^{\langle d\rangle}(\mathcal{A})_\bullet$ of $\mathcal{A}$ (introduced by Hesselholt and Madsen [HM15] in the case $d = 2$ and by Dyckerhoff [Dyc17] and Poguntke [Pog17] in general) to the derived representation theory of *Iyama's d-dimensional Auslander algebras of type* $\mathbb{A}$ [Iya11]. The relevance of the simplicial object $\mathsf{S}^{\langle d\rangle}(\mathcal{A})_\bullet$ in K-theory stems from the homotopy equivalence

$$K(A) \simeq \Omega^d|\mathsf{S}^{\langle d\rangle}(\mathcal{A})_\bullet^{\simeq}|,$$

which, by letting d vary, exhibits $K(\mathcal{A})$ as a so-called connective spectrum.

In recent work with Dyckerhoff and Lekili [DJL19] we extend the above discussion by providing a d-dimensional analogue

$$\mathsf{Fun}_{\mathbf{k}}(\mathcal{W}(\mathsf{Sym}^d\mathbb{D},\Lambda_\bullet^{(d)}),\mathcal{A})\simeq\mathsf{Fun}_{\mathbf{k}}(\mathsf{perf}A_{\bullet,d},\mathcal{A})$$

of the quasi-equivalence (a) above, induced by quasi-equivalences

$$\mathcal{W}(\mathsf{Sym}^d\mathbb{D},\Lambda_n^{(d)})\simeq\mathsf{perf}A_{n,d} \tag{2}$$

of triangulated A_∞-categories. In (2), the left-hand side denotes the partially wrapped Fukaya category associated to the d-fold symmetric product

$$\mathsf{Sym}^d\mathbb{D}:=\underbrace{\mathbb{D}\times\cdots\times\mathbb{D}}_{d\text{ times}}/\mathfrak{S}_d$$

equipped with the stops

$$\Lambda_n^{(d)}:=\bigcup_{p\in\Lambda_n}\{p\}\times\mathsf{Sym}^{d-1}\mathbb{D},$$

we refer the reader to [Aur10b, Aur10a] for the details of this construction. The existence of a quasi-equivalence in (2) is established by leveraging general generation results of Auroux [Aur10b, Aur10a] together with the explicit computation of the quasi-isomorphism type of the derived endomorphism algebra of an explicit set of generators of $\mathcal{W}(\mathsf{Sym}^d\mathbb{D},\Lambda_n^{(d)})$ following and idea of Lipshitz, Ozsváth and Thurston [LOT15]. In representation-theoretic terms, we construct an explicit tilting object in $\mathcal{W}(\mathsf{Sym}^d\mathbb{D},\Lambda_n^{(d)})$ whose endomorphism $\mathbf{k}$-algebra is isomorphic to $A_{n,d}$.

As an application of our results, and as a consequence of Koszul duality for augmented A_∞-categories, in [DJL19] we also establish the existence of quasi-equivalences

$$\mathcal{W}(\mathsf{Sym}^d\mathbb{D},\Lambda_n^{(d)})\simeq\mathcal{W}(\mathsf{Sym}^{n-d}\mathbb{D},\Lambda_n^{(n-d)}), \tag{3}$$

$n\geq d\geq 1$, thereby providing a symplectic proof of a result of Beckert [Bec18] concerning the derived equivalence between the $\mathbf{k}$-algebras $A_{n,d}$ and $A_{n,n-d}$ obtained by a delicate calculus of homotopy Kan extensions in stable derivators.

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