Spherical objects in higher Auslander–Reiten theory

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(joint work with Julian Külshammer)

Let k be a field and δ an integer number. Consider the graded algebra $B_\delta := k[\varepsilon]$ of dual numbers with ε placed in cohomological degree δ. We view B_δ as a differential graded algebra with vanishing differential. It is known [3] that the perfect derived category $\operatorname{perf} B_\delta$ of B_δ is a δ-Calabi–Yau triangulated category in the sense that $[\delta]\colon \operatorname{perf} B_\delta \to \operatorname{perf} B_\delta$ is a Serre functor. Moreover, an elementary application of derived Morita theory of A_∞-algebras shows that, up to equivalence of triangulated categories, this is the only (algebraic) triangulated category classically generated by an object whose graded endomorphism algebra is isomorphic to B_δ as a graded algebra [7].

Suppose now that m is an integer greater than or equal to 2. Then, $\operatorname{perf} B_m$ can be realised as the m-cluster category $\mathcal{C}_{\infty,m}$ of the category of finite dimensional representations of the path category

$$A_\infty := k(\cdots \to -1 \to 0 \to 1 \to \cdots),$$

Following [1] and [6], this category is defined as the orbit quotient of the bounded derived category $D^b(\operatorname{mod} A_\infty)$ which trivialises the action of the autoequivalence

$$\nu[-m]\colon D^b(\operatorname{mod} A_\infty) \to D^b(\operatorname{mod} A_\infty),$$

where $\nu := - \otimes^{\mathbb{L}}_{A_\infty} D(A_\infty)$ is the Serre functor of $D^b(\operatorname{mod} A_\infty)$. An elementary calculation shows that the image of any simple A_∞-module under the canonical projection functor $D^b(\operatorname{mod} A_\infty) \to \mathcal{C}_{\infty,m}$ is an m-spherical object which classically generates $\mathcal{C}_{\infty,m}$.

Let d be a positive integer and m an integer greater than or equal to 2. In our work in progress we apply a modified version of the above construction to the category $A^{(d)}_\infty$, a suitable d-dimensional analogue of A_∞ we introduced in [5]. This yields an md-Calabi–Yau triangulated category $\mathcal{C}^{(d)}_{\infty,m}$ with the following properties:

- If $d = 1$, then $\mathcal{C}^{(1)}_{\infty,m} = \mathcal{C}_{\infty,m}$.
- There exists a canonically defined weakly $d\mathbb{Z}$-cluster-tilting subcategory $\mathcal{O}^{(d)}_{\infty,m}$ of $\mathcal{C}^{(1)}_{\infty,m}$ (we expect that $\mathcal{O}^{(d)}_{\infty,m}$ is functorially finite in $\mathcal{C}^{(d)}_{\infty,m}$ so that the adjective 'weakly' can be removed).
- There exists an objects $S \in \mathcal{O}^{(d)}_{\infty,m}$ which is md-spherical when viewed as an object of $\mathcal{C}^{(d)}_{\infty,m}$ and which generates $\mathcal{O}^{(d)}_{\infty,m}$ in a suitable sense.
- In the case $m = 2$, the mutation combinatorics of $2d$-cluster-tilting objects in $\mathcal{C}^{(d)}_{\infty,2}$ which are contained in $\mathcal{O}^{(d)}_{\infty,2}$ is controlled by the triangulations of the cyclic apeirogon of dimension $2d$ (a higher dimensional analogue of the ∞-gon considered in [2] where this result is established in the case $d = 1$).

Our results build on earlier work of Oppermann and Thomas [8]. From the viewpoint of Iyama's higher Auslander–Reiten theory [4] the above properties suggest that $\mathcal{C}^{(d)}_{\infty,m}$ should be considered as a 'd-dimensional analogue' of $\operatorname{perf} B_m$.

Note that the differential graded algebra B_m can be interpreted as the graded singular cohomology algebra of the m-dimensional sphere with coefficients in k (we assume that k has characteristic zero for this). We conclude with the following question.

Question. *Is there a 'nice' differential graded category* $\mathcal{B}_m^{(d)}$ *such that* $\mathcal{B}_m^{(1)} = B_m$ *(viewing* B_m *as a differential graded category having a single object) and* $\operatorname{perf} \mathcal{B}_m^{(d)} = \mathcal{C}_{\infty,m}^{(d)}$*?*

Ideally, such a 'nice' differential graded category $\mathcal{B}_m^{(d)}$ would admit a topological interpretation analogous to that of B_m.

References

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A decomposition theorem for moduli of representations of algebras, with application to moduli of special biserial algebras

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1. Background

This is a report on two very recent projects concerning moduli spaces of representations of finite-dimensional associative algebras. We assume throughout that $\Bbbk$ is an algebraically closed field of characteristic 0. All algebras are assumed to be basic, finite-dimensional, and associative, thus of the form $A = \Bbbk Q/I$ for a quiver Q and (admissible) ideal I.

Given a quiver Q and *dimension vector* $\mathbf{d}\colon Q_0 \to \mathbb{Z}_{\geq 0}$, we study the *representation variety*

$$\operatorname{rep}_Q(\mathbf{d}) = \prod_{a \in Q_1} \operatorname{Mat}(\mathbf{d}(ta), \mathbf{d}(ha)), \tag{1}$$